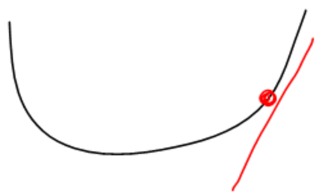
23.3 The Derivative Cont'd

$f'(x)$ represents:

the slope of the tangent line

AND the instantaneous rate of change of $f(x)$



Ex: $f(x) = x^2 + \frac{4}{x}$. Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[(x+h)^2 + \frac{4}{x+h} - \left(x^2 + \frac{4}{x}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cancel{x^2} + 2xh + h^2 + \frac{4}{x+h} - \cancel{x^2} - \frac{4}{x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2xh + h^2 + \frac{4x}{(x+h)x} - \frac{4(x+h)}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2xh + h^2 + \frac{\cancel{4x} - \cancel{4x} - 4h}{(x+h)x}\right]$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}}\left[2x + h + \frac{(-4)}{(x+h)x}\right]$$

$$= 2x + 0 - \frac{4}{x \cdot x}$$

$$= 2x - \frac{4}{x^2}$$

ASIDE

$f'(x)$ is defined for all $x \neq 0$

$f(x)$ is differentiable for all $x \neq 0$

<u>Ex</u>: Find $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. $\left(\frac{0}{0}\right)$

Multiply top and bottom by the conjugate radical.

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}\sqrt{x+h} + \cancel{\sqrt{x+h}\sqrt{x} - \sqrt{x}\sqrt{x+h}} - \sqrt{x}\sqrt{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h \cancel{- x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h} \cdot 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Ex: $f(x) = \sqrt{x+3}$. Find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h+3} - \sqrt{x+3}\right)}{h} \cdot \frac{\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}{\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{x+h+3 + \cancel{\sqrt{x+h+3}\sqrt{x+3}} - \cancel{\sqrt{x+3}\sqrt{x+h+3}} - (x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{3} - \cancel{x} - \cancel{3}}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \frac{1}{\sqrt{x+3} + \sqrt{x+3}}$$

$$= \frac{1}{2\sqrt{x+3}}$$

ASIDE

$f'(x)$ is defined for $x > -3$

$f(x)$ is differentiable for $x > -3$

NOTATION

$$f(x) = x^2 \quad \text{or} \quad y = x^2$$

The derivative $f'(x) = 2x$

or $y' = 2x$

or $\dfrac{df}{dx} = 2x$

"the derivative of f with respect to x"

or $\dfrac{dy}{dx} = 2x$

"the derivative of y with respect to x"

or $\dfrac{d}{dx}\left[x^2\right] = 2x$

"the derivative of x^2 with respect to x"

$$f'(1) = 2$$

$$y'\Big|_{x=1} = 2$$

$$\frac{dy}{dx}\Big|_{x=1} = 2$$

$$\frac{df}{dx}\Big|_{x=1} = 2$$

23.2 #15

Find m_{tan} for $\overset{f(x)}{\cancel{y}} = 1.5x^4$.

Then evaluate it at $x = 0, 0.5, 1$.

Hint: $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

This hint would be given on a test.

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1.5(x+h)^4 - 1.5x^4}{h}$$

$$= \lim_{h \to 0} \frac{1.5\left(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4\right) - 1.5x^4}{h}$$

$$\vdots$$

Full solutions are on the course website.