1. [8 marks] The DE $8y'' + 72y = 8\csc 3x$ has $y_C = C_1 \cos 3x + C_2 \sin 3x$. Solve the DE using Variation of Parameters.

1) $y_c = C_1 \cos 3x + C_2 \sin 3x$

$y_1 = \cos 3x \; , \; y_2 = \sin 3x$

2) $W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$

$= 3\cos^2 3x + 3\sin^2 3x$

$= 3(\cos^2 3x + \sin^2 3x)$

$= 3$

$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \csc 3x & \sim \end{vmatrix}$

$= -1$

$\boxed{\text{Standard Form :} \quad y'' + 9y = \underbrace{\csc 3x}_{f(x)}}$

$W_2 = \begin{vmatrix} \cos 3x & 0 \\ \sim & \csc 3x \end{vmatrix}$

$= \cos 3x \csc 3x$

$= \cot 3x$

$\longrightarrow$

3) $u_1' = \frac{W_1}{W}$

$= \frac{-1}{3}$

$u_1 = \frac{-x}{3}$

4) $u_2' = \frac{W_2}{W}$

$= \frac{1}{3} \cot 3x$

$u_2 = \frac{1}{9} \ln |\sin 3x|$

5) $y_p = u_1 y_1 + u_2 y_2$

$= \frac{-x}{3} \cos 3x + \frac{1}{9} (\sin 3x) \ln |\sin 3x|$

6) $y = y_c + y_p$

$y = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{3} \cos 3x + \frac{1}{9} (\sin 3x) \ln |\sin 3x|$

2. [4 marks] Solve:

a) $x^2y'' + 3xy' + y = 0$

Cauchy-Euler

$$m(m-1) + 3m + 1 = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

$$y = C_1 x^{-1} + C_2 x^{-1} \ln x$$

b) $x^2y'' + xy' + 4y = 0$

Cauchy-Euler

$$m(m-1) + m + 4 = 0$$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$$m = \pm 2i$$

$$y = C_1 \cos(2\ln x) + C_2 \sin(2\ln x)$$

3. [5 marks] A mass weighing 39.2 N stretches a spring by 70 cm. There is a damping force of magnitude β times the velocity. Find β so that the motion is critically-damped. (Use $g = 9.8$ N/kg.)

$$mx'' + \beta x' + kx = 0$$

$$\boxed{\frac{39.2\text{ N}}{\left(9.8\ \frac{\text{N}}{\text{kg}}\right)} = 4\text{ kg}}$$

$$\boxed{\begin{aligned} F &= kx \\ 39.2\text{ N} &= k(0.7\text{ m}) \\ 56\ \tfrac{\text{N}}{\text{m}} &= k \end{aligned}}$$

$$4x'' + \beta x' + 56x = 0$$

$$4m^2 + \beta m + 56 = 0$$

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 896}}{8}$$

Critically-damped $\Rightarrow$ repeated real roots

$$\Rightarrow \beta^2 - 896 = 0$$

$$\Rightarrow \beta^2 = 896$$

$$\Rightarrow \beta = \pm\sqrt{896}$$

$$\Rightarrow \beta = \sqrt{896} \quad (\text{since } \beta > 0)$$

4. [8 marks] Consider the DE below.

Use $y = \sum_{n=0}^{\infty} C_n x^n$ to find C_2, C_3 and C_4 in terms of C_0 and C_1.

$y'' - 7xy = 0$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$y'' - 7xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 7x \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=0}^{\infty} 7 C_n x^{n+1} = 0$$

$$\boxed{\begin{array}{c} k = n-2 \\ n = k+2 \\ n=2 \Rightarrow k=0 \end{array}} \qquad \boxed{\begin{array}{c} k = n+1 \\ n = k-1 \\ n=0 \Rightarrow k=1 \end{array}}$$

Make both series start at $k=1$.

$$\underbrace{}_{\text{1st term}} $$

$$\underbrace{2C_2}_{} + \sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^k - \sum_{k=1}^{\infty} 7 C_{k-1} x^k = 0$$

$\longrightarrow$

$$2C_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - 7C_{k-1} \right] x^k = 0$$

Set all coefficients $= 0$:

$$2C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$(k+2)(k+1)C_{k+2} - 7C_{k-1} = 0 \Rightarrow C_{k+2} = \frac{7C_{k-1}}{(k+2)(k+1)}$$

for $k \geq 1$

Sub $k=1$: $\boxed{C_3 = \frac{7C_0}{6}}$

Sub $k=2$: $\boxed{C_4 = \frac{7C_1}{12}}$