1. [3 marks] A large tank initially contains 12 L of pure water. A 2 g/L sugar solution is pumped in at a rate of 7 L/min. The well-mixed solution is pumped out at a rate of 5 L/min. Let m represent the mass of sugar in the tank (in grams) after t minutes. Write down a differential equation that relates m and t. **Do not solve the DE.**

Volume $= 12 + 2t$

$\frac{dV}{dt} = 7 \frac{L}{min} - 5 \frac{L}{min}$

$= 2 \frac{L}{min}$

$V = 2t + C_1$

$V = 12$, $t = 0$: $12 = C_1$

$V = 2t + 12$

$\frac{dm}{dt} = 2 \frac{g}{L} \cdot 7 \frac{L}{min} - \frac{m}{12+2t} \frac{g}{L} \cdot 5 \frac{L}{min}$

$$\boxed{\frac{dm}{dt} = 14 - \frac{5m}{12+2t}}$$

2. [3 marks] Find a second linearly independent solution to the following DE, given that $y_1 = \sin x^2$ is a solution:

$y'' - \frac{1}{x}y' + 4x^2 y = 0$

$P(x) = -\frac{1}{x}$

$\int P(x)dx = -\ln|x|$

$-\int P(x)dx = \ln|x|$

$e^{-\int P(x)dx} = e^{\ln|x|}$

$= |x|$

$= x \qquad (x > 0)$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$= \sin x^2 \int \frac{x}{\sin^2 x^2}\, dx$$

$$= \sin x^2 \int x \csc^2 x^2\, dx$$

Sub $u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$\int x \csc^2 x^2\, dx = \frac{1}{2}\int \csc^2 u\, du$

$= -\frac{1}{2}\cot u + C_1$

$= \sin x^2 \left(-\frac{1}{2}\cot x^2\right)$

$= -\frac{1}{2}\cos x^2$

or $y_2 = \cos x^2$

Recall: y_p has unknown coefficients until the DE is solved.

3. [4 marks] State y_C and y_P:

a) $y'' - 8y' + 25y = 3\sin 3x$

$m^2 - 8m + 25 = 0$

$m = \frac{8 \pm \sqrt{64 - 100}}{2} = \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm 6i}{2} = 4 \pm 3i$

$y_c = e^{4x}(C_1 \cos 3x + C_2 \sin 3x)$

$y_p = A\sin 3x + B\cos 3x$ (not bad case)

b) $y'' + 16y = 16\cos 4x$

$m^2 + 16 = 0$

$m^2 = -16$

$m = \pm\sqrt{-16} = \pm 4i = 0 \pm 4i$

$y_c = e^{0x}(C_1 \cos 4x + C_2 \sin 4x)$

$y_c = C_1 \cos 4x + C_2 \sin 4x$

$y_p = Ax\cos 4x + Bx\sin 4x$ (bad case)

4. [4 marks] Solve:
$y'' - 4y' + 4y = 0, \quad y(1) = 3e^2, \quad y'(1) = 10e^2$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2 \Rightarrow y_1 = e^{2x}, \; y_2 = xe^{2x}$

$$\boxed{y = C_1 e^{2x} + C_2 x e^{2x}}$$

$y = 3e^2$, $x = 1$: $3e^2 = C_1 e^2 + C_2 e^2$

$3 = C_1 + C_2$ (1)

$y' = 2C_1 e^{2x} + C_2(2xe^{2x} + e^{2x})$

$y' = 10e^2$, $x = 1$: $10e^2 = 2C_1 e^2 + C_2(2e^2 + e^2)$

$10e^2 = 2C_1 e^2 + 3C_2 e^2$

$10 = 2C_1 + 3C_2$ (2)

$-2 \times (1)$: $-6 = -2C_1 - 2C_2$

(2): $10 = 2C_1 + 3C_2$

$+$

$4 = C_2$

$C_2 = 4 \rightarrow (1)$: $3 = C_1 + 4 \Rightarrow C_1 = -1$

$$\boxed{y = (-1 + 4x)e^{2x}}$$

5. [6 marks] Solve $y'' - 2y' - 15y = 2e^{4x} + 4x$.

$m^2 - 2m - 15 = 0$

$(m-5)(m+3) = 0$

$m = 5, -3$

$y_c = C_1 e^{5x} + C_2 e^{-3x}$

$y_p = Ae^{4x} + Bx + C$ (not bad case)

$$\begin{cases} y_p = Ae^{4x} + Bx + C \\ y_p' = 4Ae^{4x} + B \\ y_p'' = 16Ae^{4x} \end{cases} \rightarrow DE$$

$y'' - 2y' - 15y = 2e^{4x} + 4x$

$16Ae^{4x} - 2(4Ae^{4x} + B) - 15(Ae^{4x} + Bx + C) = 2e^{4x} + 4x$

$16Ae^{4x} - 8Ae^{4x} - 2B - 15Ae^{4x} - 15Bx - 15C = 2e^{4x} + 4x$

$-7Ae^{4x} - 15Bx + (-2B - 15C) = 2e^{4x} + 4x + 0$

$-7A = 2 \Rightarrow A = -\frac{2}{7}$

$-15B = 4 \Rightarrow B = -\frac{4}{15}$

$-2B - 15C = 0 \Rightarrow \frac{8}{15} - 15C = 0 \Rightarrow C = \frac{8}{225}$

$y = y_c + y_p$

$$\boxed{y = C_1 e^{5x} + C_2 e^{-3x} - \frac{2}{7}e^{4x} - \frac{4}{15}x + \frac{8}{225}}$$