1. Solve using the Laplace transform:
$y'' - 10y' + 24y = 2e^{4t}, y(0) = 8, y'(0) = -3$

1) Apply $\mathcal{L}$

$$s^2Y(s) - \cancel{s}y(0) - y'(0) - 10[sY(s) - y(0)] + 24Y(s) = \frac{2}{s-4}$$

$$s^2Y(s) - 8s + 3 - 10[sY(s) - 8] + 24Y(s) = \frac{2}{s-4}$$

2) Solve for $Y(s)$

$$s^2Y(s) - 8s + 3 - 10sY(s) + 80 + 24Y(s) = \frac{2}{s-4}$$

$$(s^2 - 10s + 24)Y(s) = 8s - 83 + \frac{2}{s-4}$$

$$Y(s) = \frac{8s-83}{s^2-10s+24} + \frac{2}{(s-4)(s^2-10s+24)}$$

$$Y(s) = \frac{(8s-83)(s-4)+2}{(s-4)(s^2-10s+24)}$$

$$Y(s) = \frac{8s^2-115s+334}{(s-4)^2(s-6)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{8s^2-115s+334}{(s-4)^2(s-6)}\right\} \quad (*)$$

4) Partial Fractions

$$\frac{8s^2-115s+334}{(s-4)^2(s-6)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{s-6} \rightarrow$$

$$8s^2 - 115s + 334 = A(s-4)(s-6) + B(s-6) + C(s-4)^2$$

Sub $s=4$: $\quad 2 = -2B \Rightarrow B = -1$

$s=6$: $\quad -68 = 4C \Rightarrow C = -17$

s^2 coefficient : $\quad 8 = A + C \Rightarrow A = 25$

5) from ⊛

$$y(t) = \mathcal{L}^{-1}\left\{\frac{25}{s-4} - \frac{1}{(s-4)^2} - \frac{17}{s-6}\right\}$$

$$= 25e^{4t} - te^{4t} - 17e^{6t}$$

2. Solve using the Laplace transform:
$y'' - 3y' = 10e^t \sin t, y(0) = 0, y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] = \frac{10}{(s-1)^2+1}$$

$$s^2Y(s) - 3sY(s) = \frac{10}{s^2-2s+2}$$

2) Solve for $Y(s)$

$$(s^2-3s)Y(s) = \frac{10}{s^2-2s+2}$$

$$Y(s) = \frac{10}{s(s-3)(s^2-2s+2)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{10}{s(s-3)(s^2-2s+2)}\right\} \quad *$$

4) Partial Fractions

$$\frac{10}{s(s-3)(s^2-2s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{Cs+D}{s^2-2s+2}$$

$$10 = A(s-3)(s^2-2s+2) + Bs(s^2-2s+2) + (Cs+D)s(s-3)$$

Sub $s=0$: $10 = -6A \Rightarrow A = -5/3$

Sub $s=3$: $10 = 15B \Rightarrow B = 2/3$

s^3 coefficient: $0 = A+B+C \Rightarrow C = 1$

Sub $s=1$: $10 = -2A + B + (C+D)(-2)$

$10 = -2A + B - 2C - 2D$

$D = -4$

$\rightarrow$

5) From ⊛ : $y(t) = \mathcal{L}^{-1}\left\{ -\frac{5}{3} \cdot \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-4}{s^2-2s+2} \right\}$

$= \mathcal{L}^{-1}\left\{ -\frac{5}{3} \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-4}{(s-1)^2+1} \right\}$

$$\boxed{\begin{array}{l} \text{Need } s-4 \text{ in terms of } s-1. \\ \text{Let } s-4 = a(s-1)+b \\ \quad s-4 = s-1+b \\ \quad s-4 = (s-1)-3 \end{array}}$$

$= \mathcal{L}^{-1}\left\{ -\frac{5}{3} \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-1}{(s-1)^2+1} - 3 \cdot \frac{1}{(s-1)^2+1} \right\}$

$= -\frac{5}{3} + \frac{2}{3} e^{3t} + e^{t} \cos t - 3e^{t} \sin t$

3.

a) Solve using the Laplace transform:

$y' + 8y = f(t), y(0) = 0$, where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 9, & t \ge 3 \end{cases}$

b) Find $y(1)$

c) Find $y(4)$

a)

1) Apply $\mathcal{L}$

$$sY(s) - y(0) + 8Y(s) = \mathcal{L}\{9\mathcal{U}(t-3)\} \leftarrow \boxed{e^{-3s}\mathcal{L}\{9\} = \frac{e^{-3s}\,9}{s}}$$

$$sY(s) + 8Y(s) = \frac{9e^{-3s}}{s}$$

2) Solve for $Y(s)$

$$(s+8)Y(s) = \frac{9e^{-3s}}{s}$$

$$Y(s) = \frac{9e^{-3s}}{s(s+8)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{9}{s(s+8)}e^{-3s}\right\} \quad (*)$$

4) Partial Fractions

$$\frac{9}{s(s+8)} = \frac{A}{s} + \frac{B}{s+8}$$

$$9 = A(s+8) + Bs$$

$s=0$: $\quad 9 = 8A \Rightarrow A = 9/8$

$s=-8$: $\quad 9 = -8B \Rightarrow B = -9/8$

5) From $(*)$

$$y(t) = \mathcal{L}^{-1}\left\{\left[\frac{9}{8}\frac{1}{s} - \frac{9}{8}\frac{1}{s+8}\right]e^{-3s}\right\} \rightarrow$$

$$y(t) = f(t-3)\,u(t-3)$$

$$\boxed{\begin{aligned} F(s) &= \frac{9}{8}\frac{1}{s} - \frac{9}{8}\frac{1}{s+8} \\ f(t) &= \frac{9}{8} - \frac{9}{8}e^{-8t} \\ f(t-3) &= \frac{9}{8} - \frac{9}{8}e^{-8(t-3)} \end{aligned}}$$

$$y(t) = \left[\frac{9}{8} - \frac{9}{8}e^{-8(t-3)}\right]u(t-3)$$

or $\begin{cases} 0, & 0 \le t < 3 \\ \frac{9}{8} - \frac{9}{8}e^{-8(t-3)}, & t \ge 3 \end{cases}$

b) $y(1) = 0$

c) $y(4) = \frac{9}{8} - \frac{9}{8}e^{-8}$

4. Solve for $f(t)$:

$f(t) = t^2 - \int_0^t f(\theta)e^{t-\theta}d\theta$

$$f(t) = t^2 - f * e^t$$

Apply $\mathcal{L}$:

$$F(s) = \frac{2}{s^3} - F(s)\cdot\frac{1}{s-1}$$

Solve for $F(s)$:

$$F(s) + F(s)\frac{1}{s-1} = \frac{2}{s^3}$$

$$\left[1 + \frac{1}{s-1}\right]F(s) = \frac{2}{s^3}$$

$$\frac{s}{s-1}F(s) = \frac{2}{s^3}$$

$$F(s) = \frac{2(s-1)}{s^4}$$

Apply $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^3} - \frac{2}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2!}{s^3} - \frac{1}{3}\frac{3!}{s^4}\right\}$$

$$= t^2 - \frac{t^3}{3}$$