(1)

1) $y_c = C_1 e^{3x} + C_2 e^{4x}$

$y_1 = e^{3x}, \quad y_2 = e^{4x}$

2) $$W = \begin{vmatrix} e^{3x} & e^{4x} \\ 3e^{3x} & 4e^{4x} \end{vmatrix}$$

$$= 4e^{7x} - 3e^{7x}$$

$$= e^{7x}$$

$$W_1 = \begin{vmatrix} 0 & e^{4x} \\ 8x^3 & 4e^{4x} \end{vmatrix}$$

$\boxed{\text{DE is in standard form } \checkmark}$

$$= -8x^3 e^{4x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 8x^3 \end{vmatrix}$$

$$= 8x^3 e^{3x}$$

3) $$u_1' = \frac{W_1}{W}$$

$$= -8x^3 e^{-3x}$$

	D	I
$+$	$-8x^3$	e^{-3x}
$-$	$-24x^2$	$-\frac{1}{3} e^{-3x}$
$+$	$-48x$	$\frac{1}{9} e^{-3x}$
$-$	-48	$-\frac{1}{27} e^{-3x}$
		$\frac{1}{81} e^{-3x}$

$$u_1 = \left(\frac{8x^3}{3} + \frac{24x^2}{9} + \frac{48x}{27} + \frac{48}{81}\right) e^{-3x}$$

4) $u_2' = \frac{W_2}{W}$

$= 8x^3 e^{-4x}$

	D	I
$+$	$8x^3$	e^{-4x}
$-$	$24x^2$	$-\frac{1}{4} e^{-4x}$
$+$	$48x$	$\frac{1}{16} e^{-4x}$
$-$	48	$-\frac{1}{64} e^{-4x}$
		$\frac{1}{256} e^{-4x}$

$$u_2 = \left(\frac{-8x^3}{4} - \frac{24x^2}{16} - \frac{48x}{64} - \frac{48}{256}\right)e^{-4x}$$

5) $y_p = u_1 y_1 + u_2 y_2$

$$= \frac{8x^3}{3} + \frac{24x^2}{9} + \frac{48x}{27} + \frac{48}{81}$$

$$\frac{-8x^3}{4} - \frac{24x^2}{16} - \frac{48x}{64} - \frac{48}{256}$$

$$= \frac{2x^3}{3} + \frac{7x^2}{6} + \frac{37x}{36} + \frac{175}{432}$$

6) $y = y_c + y_p$

$$y = C_1 e^{3x} + C_2 e^{4x} + \frac{2x^3}{3} + \frac{7x^2}{6} + \frac{37x}{36} + \frac{175}{432}$$

(2) a) Cauchy-Euler

$$m(m-1) + 4m + 13 = 0$$

$$m^2 + 3m + 13 = 0$$

$$m = \frac{-3 \pm \sqrt{-43}}{2}$$

$$m = -\frac{3}{2} \pm \frac{\sqrt{43}}{2} i$$

$$y = x^{-3/2}\left[C_1 \cos\left(\frac{\sqrt{43}}{2}\ln x\right) + C_2 \sin\left(\frac{\sqrt{43}}{2}\ln x\right)\right]$$

b) Cauchy-Euler

$$m(m-1) + 9m + 16 = 0$$

$$m^2 + 8m + 16 = 0$$

$$(m+4)^2 = 0$$

$$m = -4, -4$$

$$y = C_1 x^{-4} + C_2 x^{-4} \ln x$$

→

c) Cauchy-Euler

$$m(m-1) - 3m + 2 = 0$$

$$m^2 - 4m + 2 = 0$$

$$m = \frac{4 \pm \sqrt{8}}{2}$$

$$m = \frac{4 \pm 2\sqrt{2}}{2}$$

$$m = 2 \pm \sqrt{2}$$

$$m = 2+\sqrt{2},\ 2-\sqrt{2}$$

(distinct real roots)

$$y = C_1 x^{2+\sqrt{2}} + C_2 x^{2-\sqrt{2}}$$

③ a) $m x'' + \beta x' + k x = 0$

mass, in kg

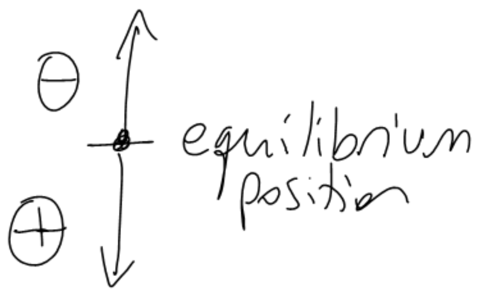
magnitude of damping force, as a multiple of velocity

spring constant, in $\frac{N}{m}$

$$3x'' + 27x = 0$$

$$\boxed{x'' + 9x = 0}$$

$$\boxed{x(0) = 0.2,\ x'(0) = -0.9}$$



Auxiliary equation:

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm\sqrt{-9}$$

$$m = \pm 3i$$

$$\boxed{x = C_1 \cos 3t + C_2 \sin 3t}$$

$x = 0.2$, $t = 0$: $0.2 = C_1$

$$\boxed{x = 0.2\cos 3t + C_2 \sin 3t}$$

$$x' = -0.6\sin 3t + 3C_2\cos 3t$$

$x' = -0.9$, $t = 0$: $\quad -0.9 = 3C_2$

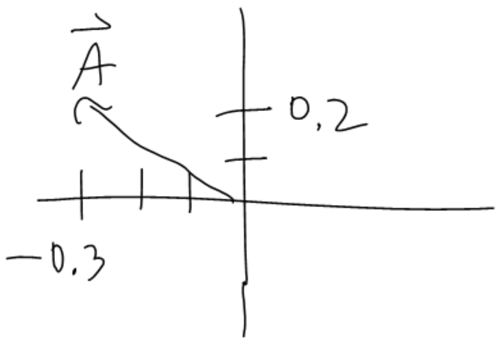
$$-0.3 = C_2$$

$$\boxed{x = 0.2\cos 3t - 0.3\sin 3t}$$

b)

$$A\cos\phi = -0.3$$

$$A\sin\phi = 0.2$$



$$A = \sqrt{0.2^2 + (-0.3)^2} = \sqrt{0.13}$$

$$\phi = \tan^{-1}\left(\frac{0.2}{-0.3}\right) \quad (+\pi?)$$

$$= -\tan^{-1}\left(\frac{2}{3}\right) + \pi$$

$$= \pi - \tan^{-1}\frac{2}{3}$$

$$x = \sqrt{0.13}\sin\left(3t + \pi - \tan^{-1}\frac{2}{3}\right)$$

c) Lowest point

$\Rightarrow x$ is maximized

$\Rightarrow \sin\left(3t+\pi-\tan^{-1}\frac{2}{3}\right)=1$

$\Rightarrow \quad 3t+\pi-\tan^{-1}\frac{2}{3}=\frac{\pi}{2},\ \frac{5\pi}{2},\ \frac{9\pi}{2},\ldots$

$3t+\pi-\tan^{-1}\frac{2}{3}=\frac{\pi}{2} \Rightarrow t\approx -0.33\text{ s}$

Nonsense

$3t+\pi-\tan^{-1}\frac{2}{3}=\frac{5\pi}{2} \Rightarrow t\approx 1.77\text{ s}$

(4) a) $mx'' + \beta x' + kx = 0$

mass, in kg

magnitude of damping force, as a multiple of velocity

spring constant, in $\frac{N}{m}$

$$\boxed{8x'' + 4x' + 3x = 0}$$

b) $8m^2 + 4m + 3 = 0$

$$m = \frac{-4 \pm \sqrt{-80}}{16}$$

Complex roots $\Rightarrow$ underdamped motion

<u>Aside</u>

distinct real roots $\Rightarrow$ overdamped motion

repeated real roots $\Rightarrow$ critically-damped motion