

1. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Write $\mathbf{w} = \begin{bmatrix} 93 \\ -61 \\ 87 \end{bmatrix}$ as a linear combination of the basis vectors.

Orthogonal basis

$$\Rightarrow \bar{\mathbf{w}} = \text{proj}_{\bar{\mathbf{v}}_1} \bar{\mathbf{w}} + \text{proj}_{\bar{\mathbf{v}}_2} \bar{\mathbf{w}} + \text{proj}_{\bar{\mathbf{v}}_3} \bar{\mathbf{w}}$$

$$= \frac{\bar{\mathbf{v}}_1 \cdot \bar{\mathbf{w}}}{\|\bar{\mathbf{v}}_1\|^2} \bar{\mathbf{v}}_1 + \dots$$

$$= \frac{145}{9} \bar{\mathbf{v}}_1 - \frac{49}{9} \bar{\mathbf{v}}_2 + \frac{395}{9} \bar{\mathbf{v}}_3$$

2.

Find an orthogonal basis for span

$$\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right).$$

Gram-Schmidt orthogonalization

$$\text{Partial Basis } X = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$3\vec{v}_2 = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix}$$

$$\text{Partial Basis } X = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix} \right\}$$

$$\vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix}} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{36}{123} \right) \begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix}$$

$$\left(\frac{12}{41} \right)$$

$$41\vec{v}_3 = 41 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} - 41 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - 12 \begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 87 \\ 27 \\ -84 \end{bmatrix}$$

$$\text{Orthogonal Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \\ 7 \end{bmatrix}, \begin{bmatrix} 11 \\ 87 \\ 27 \\ -84 \end{bmatrix} \right\}$$

3.

$$\text{Let } W = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 7 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -11 \\ 2 \\ -1 \end{bmatrix}\right).$$

Find a basis for $W^\perp$.

$$\text{Solve } \left[\begin{array}{ccccc|c} 1 & -3 & 5 & 0 & 1 & 0 \\ -1 & 4 & 7 & 1 & 2 & 0 \\ 0 & -1 & -11 & 2 & -1 & 0 \end{array} \right]$$

$$R_2 + R_1 \quad \left[\begin{array}{ccccc|c} 1 & -3 & 5 & 0 & 1 & 0 \\ 0 & 1 & 12 & 1 & 3 & 0 \\ 0 & -1 & -11 & 2 & -1 & 0 \end{array} \right]$$

$$R_1 + 3R_2 \quad \left[\begin{array}{ccccc|c} 1 & 0 & 41 & 3 & 10 & 0 \\ 0 & 1 & 12 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array} \right]$$

$$R_3 + R_2 \quad \left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 0 & 0 & -12 & -72 & 0 \\ 0 & 1 & 0 & -35 & -21 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array} \right]$$

$$R_1 - 41R_3$$

$$R_2 - 12R_3$$

$$\begin{array}{c} \uparrow \quad \uparrow \\ x_4 = s \quad x_5 = t \end{array}$$

$$x_1 = 120s + 72t$$

$$x_2 = 35s + 21t$$

$$x_3 = -3s - 2t$$

$$\vec{x} = \begin{bmatrix} 120 \\ 35 \\ -3 \\ 0 \end{bmatrix} s + \begin{bmatrix} 72 \\ 21 \\ -2 \\ 0 \end{bmatrix} t$$

$$\text{A basis for } W^\perp \text{ is } \left\{ \begin{bmatrix} 120 \\ 35 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 72 \\ 21 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

4. The matrix A has eigenvalue $\lambda_1 = 2$ corresponding to the eigenvector $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and eigenvalue $\lambda_2 = 3$ corresponding to the eigenvector $\vec{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find $A^3 \begin{bmatrix} 22 \\ 23 \end{bmatrix}$ using the formula $A^n(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\lambda_1^n\vec{x}_1 + c_2\lambda_2^n\vec{x}_2$.

$$\text{Let } c_1\vec{x}_1 + c_2\vec{x}_2 = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

$$\begin{array}{cc|c} c_1 & c_2 & \\ \hline 3 & 2 & 22 \\ 2 & 3 & 23 \end{array}$$

$$\frac{R_1}{3} \quad \begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ \hline 2 & 3 & 23 \end{array}$$

$$R_2 - 2R_1 \quad \begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ \hline 0 & \frac{5}{3} & \frac{25}{3} \end{array}$$

$$\frac{3}{5} \times R_2 \quad \begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ \hline 0 & 1 & 5 \end{array}$$

$$R_1 - \frac{2}{3}R_2 \quad \begin{array}{cc|c} 1 & 0 & 4 \\ \hline 0 & 1 & 5 \end{array}$$

$$c_1 = 4, c_2 = 5$$

$$A^3 \begin{bmatrix} 22 \\ 23 \end{bmatrix} = A^3 \left(4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

$$= c_1 \lambda_1^3 \vec{x}_1 + c_2 \lambda_2^3 \vec{x}_2$$

$$= 4(2^3) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5(3^3) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 32 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 135 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 366 \\ 469 \end{bmatrix}$$

5. The matrix A has the eigenvalue 2 corresponding to the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and the eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find A^n . Simplify your answer as much as possible.

$$P^{-1}AP = D$$

$$\cancel{P}P^{-1}AP = PD$$

$$A\cancel{P}P^{-1} = PDP^{-1}$$

$$A^n = \cancel{P}P^{-1}\cancel{P}P^{-1}\dots\cancel{P}P^{-1}$$

$$\boxed{A^n = PD^nP^{-1}}$$

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad P^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$$

$$A^n = PD^nP^{-1}$$

$$= \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 3 \cdot 2^n & 2 \cdot 3^n \\ 2 \cdot 2^n & 3 \cdot 3^n \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 9 \cdot 2^n - 4 \cdot 3^n & 2 \cdot 3^{n+1} - 3 \cdot 2^{n+1} \\ 3 \cdot 2^{n+1} - 2 \cdot 3^{n+1} & 3^{n+2} - 2^{n+2} \end{bmatrix}$$