8.3 Future Value of an Annuity

Definition: An **annuity** is a sequence of equal, periodic payments that earn compound interest. One example is a set of regular monthly savings towards retirement.

Fact: The formula for the future value of an annuity is

$\text{FV} = \text{PMT} \cdot \frac{((1+i)^n - 1)}{i}$, where

FV is the future value, in dollars
PMT is the periodic payment, in dollars
$i = \frac{r}{m}$ is the interest rate per compounding period
$n = mt$ is the total number of compounding periods

Comment: Recall that:
r is the annual nominal interest rate, expressed as a decimal
m is the number of compounding periods per year
t is the time, in years

Comment: We are making two assumptions about all the annuities in this course. The payment period will always be the same as the compounding period. Payments will always be at the end of each compounding period.

Example: We make semi-annual deposits of \$1500 for 30 years. Our account pays 5%, compounded semi-annually. What is the value of the investment after 30 years? How much is interest?

$PMT = 1500 \quad t = 30 \quad r = 0.05 \quad m = 2 \quad FV = ?$

$i = \frac{r}{m} = 0.025$

$n = mt = 60$

$$FV = PMT \cdot \frac{((1+i)^n - 1)}{i}$$

$$FV = 1500 \cdot \frac{(1.025^{60} - 1)}{0.025}$$

Can also be written

$$FV = \frac{1500\,(1.025^{60} - 1)}{0.025}$$

$FV \approx \$\,203,987.38$

$$\text{Interest} = FV - \text{Total Payments}$$
$$\approx 203,987.38 - 60(1500)$$
$$\approx \$\,113,987.38$$

Definition: A **sinking fund** is an investment aiming to save a specified amount of money.

Example: Quick Algebra Review
Solve for x in the equation $9 = x \cdot \frac{3}{4}$

Multiply both sides by $\frac{4}{3}$:

$$9 \cdot \frac{4}{3} = x \cdot \frac{3}{4} \cdot \frac{4}{3}$$

$$12 = x$$

Example: A company needs to replace a $\$600,000$ piece of equipment in seven years. They make quarterly payments into an account paying 9%, compounded quarterly. How much should each payment be?

$FV = 600{,}000 \quad t = 7 \quad m = 4 \quad r = 0.09 \quad PMT = ?$

$$i = \frac{r}{m} = 0.0225$$

$$n = mt = 28$$

$$FV = PMT \cdot \frac{((1+i)^n - 1)}{i}$$

$$600{,}000 = PMT \cdot \frac{(1.0225^{28} - 1)}{0.0225}$$

$$600{,}000 \cdot \frac{0.0225}{(1.0225^{28} - 1)} = PMT$$

$$PMT \approx \$15{,}615.15$$