8.2 Compound Interest

In this section we'll study one-time investments that earn compound interest.

Definition: An investment earns **compound interest** if the interest is reinvested at the end of each compounding period.

Fact: The formula for a one-time investment earning compound interest is:

$A = P(1 + \frac{r}{m})^{mt}$, where:

A is the future value, in dollars
P is the present value, in dollars (sometimes called the principal)
r is the annual nominal interest rate
m is the number of compounding periods per year
t is the time, in years

Comment: The annual nominal interest rate r is expressed as a decimal.
So 3% annual interest means $r = 0.03$

Example: Calculate the future value and the amount of interest if \$100 is invested for ten years at 5% compounded:

a) annually

$A = ?$ $\quad P = 100 \quad t = 10 \quad r = 0.05 \quad m = 1$

$$A = P(1 + \frac{r}{m})^{mt}$$

$$A = 100(1 + 0.05)^{10}$$

$$A \approx \$162.89$$

Interest $I = A - P$

$$\approx \$62.89$$

Note: Use $\boxed{y^x}$ for exponents on Sharp and BA

b) monthly

Same as above, but $m=12$

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

$$A = 100\left(1+\frac{0.05}{12}\right)^{120}$$

$$A \approx \$164.70$$

Interest $I = A - P$

$$\approx \$64.70$$

Definition: **Euler's number** is written e and it has the value $e \approx 2.718$

Example: Here are the keystrokes for finding e on the Sharp and BA calculators.

Sharp: $\boxed{2^{nd}F}$ $\boxed{\ln}$ (e^x) $\boxed{1}$ $\boxed{=}$

BA: $\boxed{1}$ $\boxed{2ND}$ $\boxed{LN}$ (e^x)

Fact: Recall that m is the number of compounding periods per year. As m gets larger, the value of $(1+\frac{r}{m})^{mt}$ gets closer to e^{rt}.

Definition: **Continuous compounding** is a theoretical scenario where interest is constantly reinvested. It roughly describes what happens when the number of compounding periods per year gets very large.

Fact: The formula for a one-time investment undergoing continuous compounding is $A = Pe^{rt}$.

Why? $A = P\underbrace{\boxed{\left(1+\frac{r}{m}\right)^{mt}}}_{e^{rt}}$

for large m

Example: Calculate the future value and the amount of interest if \$100 is invested for two years at 18% compounded:

a) continuously

$P = 100 \quad t = 2 \quad r = 0.18$

$A = Pe^{rt}$

$A = 100\, e^{0.36}$

$A \approx \$143.33$

Sharp: $\boxed{100}\ \boxed{\times}\ \boxed{2^{nd}F}\ \overset{e^x}{\boxed{\ln}}\ \boxed{0.36}\ \boxed{=}$

BA: $\boxed{0.36}\ \boxed{2ND}\ \overset{e^x}{\boxed{LN}}\ \boxed{\times}\ \boxed{100}$

Interest $I = A - P$

$I \approx \$43.33$

b) daily

Same as above but $m = 365$

$A = P\left(1 + \frac{r}{m}\right)^{mt}$

$A = 100\left(1 + \frac{0.18}{365}\right)^{730}$

$A \approx \$143.32$

Interest $I = A - P$

$\approx \$43.32$

Example: How much should you invest today at 6% in order to have $\$10,000$ in five years if interest is compounded:

a) quarterly?

$P = ?$ $\quad r = 0.06$ $\quad A = 10,000$ $\quad t = 5$ $\quad m = 4$

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$10,000 = P(1 + 0.015)^{20}$$

$$\frac{10,000}{(1+0.015)^{20}} = P$$

$$P \approx \$\,7424.70$$

b) continuously?

Now use $A = Pe^{rt}$ (no m)

$$10,000 = Pe^{0.3}$$

$$\frac{10,000}{e^{0.3}} = P$$

$$P \approx \$\,7408.18$$

BA: $\boxed{10000}$ $\boxed{\div}$ $\boxed{0.3}$ $\boxed{2ND}$ $\overset{e^x}{\boxed{LN}}$ $\boxed{=}$

Definition: The **annual percentage yield**, written APY, is the rate that is actually paid. It's sometimes called the effective rate.

Fact: The APY for an investment earning compound interest is APY$= (1 + \frac{r}{m})^m - 1$.
The APY for an investment undergoing continuous compounding is APY$= e^r - 1$.

Example: Calculate the annual percentage yield for:

a) 4.95% compounded quarterly

$r = 0.0495 \quad m = 4$

$APY = \left(1 + \frac{0.0495}{4}\right)^4 - 1$

≈ 0.0504

or 5.04%

b) 3.8% compounded continuously

$r = 0.038$

$APY = e^r - 1$

$= e^{0.038} - 1$

≈ 0.0387

or 3.87%