7.2 The Stationary Matrix

Definition: The **stationary matrix** is written S. It is a state matrix that has the property that $SP = S$.

Fact: If S exists then $S_k \approx S$ for large values of k.

Example: Let $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$.

a) Check that $S = \begin{bmatrix} 0.625 & 0.375 \end{bmatrix}$ and interpret S.

Check that $SP = S$.

$$SP = \begin{bmatrix} 0.625 & 0.375 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.625 & 0.375 \end{bmatrix}$$

$$= S \checkmark$$

Once we reach state matrix S, all future state matrices will be S.

b) What will S_{50} look like?

S

c) In the distant future, what will the state matrix look like?

S

Comment: If S exists then the Markov chain is sometimes called a **regular Markov chain**.

Example: Let $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$. Find the stationary matrix.

Let $S = [x \quad y]$

Solve $SP = S$ together with $x + y = 1$.

$SP = S$: $\quad [x \quad y]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = [x \quad y]$

$$[0.9x + 0.2y \qquad 0.1x + 0.8y] = [x \quad y]$$

$$\begin{aligned} 0.9x + 0.2y &= x \quad \Longrightarrow \quad -0.1x + 0.2y = 0 \\ 0.1x + 0.8y &= y \quad \Longrightarrow \quad 0.1x - 0.2y = 0 \\ & \qquad\qquad\qquad\quad\; x + y = 1 \end{aligned}$$

$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0.1 & -0.2 & 0 \\ -0.1 & 0.2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 - 0.1R_1 \\ R_3 + 0.1R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -0.3 & -0.1 \\ 0 & 0.3 & 0.1 \end{array}\right]$$

Example Continued...

$$\frac{R_2}{-0.3}\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0.3 & 0.1 \end{array}\right] \leftarrow \frac{-0.1}{-0.3} \times \frac{10}{10} = \frac{-1}{-3} = \frac{1}{3}$$

$$\begin{array}{r} R_1 - R_2 \\ \\ R_3 - 0.3R_2 \end{array} \quad \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \leftarrow 1 - \frac{1}{3} \\ \\ \leftarrow 0.1 - 0.3\left(\frac{1}{3}\right) \end{array}$$

$$x = \frac{2}{3}$$

$$y = \frac{1}{3}$$

bottom row: no info

$$S = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Example: Let $P = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 0.7 \end{bmatrix}$. Find the stationary matrix.

Let $S = [x \quad y \quad z]$

Solve $SP = S$ together with $x+y+z=1$

$SP=S$: $\quad [x \ y \ z] \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 0.7 \end{bmatrix} = [x \ y \ z]$

$$[0.7x + 0.5y + 0.3z \quad 0.5y \quad 0.3x + 0.7z] = [x \ y \ z]$$

$$0.7x + 0.5y + 0.3z = x \Rightarrow -0.3x + 0.5y + 0.3z = 0$$

$$0.5y = y \Rightarrow -0.5y = 0$$

$$0.3x + 0.7z = z \Rightarrow 0.3x - 0.3z = 0$$

$$x + y + z = 1$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -0.5 & 0 & 0 \\ 0.3 & 0 & -0.3 & 0 \\ -0.3 & 0.5 & 0.3 & 0 \end{array}\right]$$

Example Continued...

$$\begin{array}{l} \\ \\ R_3-0.3R_1 \\ R_4+0.3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -0.5 & 0 & 0 \\ 0 & -0.3 & -0.6 & -0.3 \\ 0 & 0.8 & 0.6 & 0.3 \end{array}\right]$$

$$\frac{R_2}{-0.5} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -0.3 & -0.6 & -0.3 \\ 0 & 0.8 & 0.6 & 0.3 \end{array}\right]$$

$$\begin{array}{l} R_1-R_2 \\ \\ R_3+0.3R_2 \\ R_4-0.8R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.6 & -0.3 \\ 0 & 0 & 0.6 & 0.3 \end{array}\right]$$

$$\frac{R_3}{-0.6} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0.6 & 0.3 \end{array}\right] \leftarrow \frac{-0.3}{-0.6} \times \frac{10}{10} = \frac{-3}{-6} = \frac{1}{2}$$

$$\begin{array}{l} \\ R_1-R_3 \\ \\ \\ R_4-0.6R_3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\boxed{x = \tfrac{1}{2},\ y = 0,\ z = \tfrac{1}{2}} \implies S = \begin{bmatrix} \tfrac{1}{2} & 0 & \tfrac{1}{2} \end{bmatrix}$$