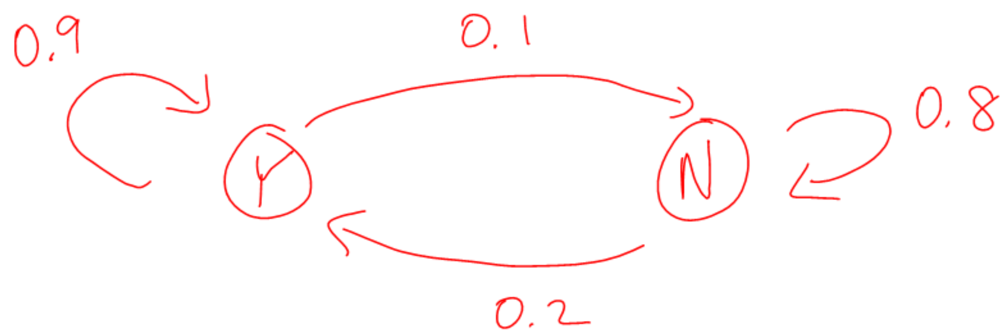
7.1 Markov Chains

In this chapter we'll use matrices to make predictions.

Example: We have the following information about CleanHair Shampoo company:
90% of CleanHair customers will buy CleanHair next time.
10% of CleanHair customers won't buy CleanHair next time.
20% of other brand customers will buy CleanHair next time.
80% of other brand customers won't buy CleanHair next time.

a) Draw the transition diagram.

States : Y : a consumer uses CleanHair
N : a consumer does not use CleanHair



b) Find the transition matrix.

$$P = \begin{array}{cc} & \begin{array}{cc} Y & N \end{array} \\ \begin{array}{c} Y \\ N \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

← next state

↑ current state

Fact: Properties of a Transition Matrix:
1) It's a square matrix (2×2 or 3×3).
2) All entries must be at least zero.
3) Each **row** sums to 1.

Comment: In a transition diagram, the arrows **leaving** a state sum to 1.

Definition: The **initial state matrix** is written S_0.

Example: Interpret $S_0 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$ in the context of the CleanHair Shampoo example.

$$S_0 = \begin{array}{cc} Y & N \\ \hline \end{array}$$
$$S_0 = \begin{array}{c} \begin{matrix} Y & N \end{matrix} \\ \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \end{array}$$

States must be in same order as in P.

40% of consumers use CleanHair now.
60% " " don't use " "

Fact: Given the initial state matrix S_0 and the transition matrix P:
The next state matrix is $S_1 = S_0 P$
The state matrix after that is $S_2 = S_1 P$
The state matrix after that is $S_3 = S_2 P$
etc.

Example: Suppose $S_0 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$ and $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$, where the states are Y and N in that order. Find S_1 and S_2 and interpret them in the context of the CleanHair shampoo example.

$$S_1 = S_0 P$$
$$= \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$
$$= \begin{array}{c} \begin{array}{cc} Y & N \end{array} \\ \begin{bmatrix} 0.48 & 0.52 \end{bmatrix} \end{array}$$

48% of consumers will buy CleanHair on their next purchase.

$$S_2 = S_1 P$$
$$= \begin{bmatrix} 0.48 & 0.52 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$
$$= \begin{array}{c} \begin{array}{cc} Y & N \end{array} \\ \begin{bmatrix} 0.536 & 0.464 \end{bmatrix} \end{array}$$

53.6% of consumers will buy CleanHair 2 purchases from now.

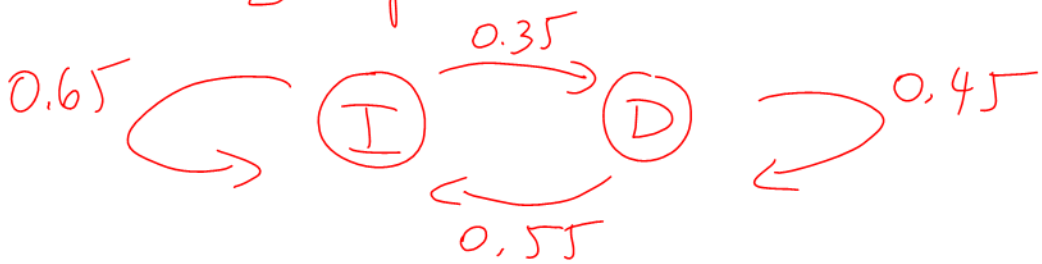
Fact: Given any state matrix (S_0, S_1, S_2 etc.), the probabilities sum to 1.

Definition: A **Markov chain** is a sequence of state matrices $S_0, S_1, S_2, S_3, \ldots$.

Example: We have the following information about the price of a certain company's stock:
If the price increases one day then the probability that it increases the next day is 65%.
If the price decreases one day then the probability that it decreases the next day is 45%.

a) Draw the transition diagram.

States: I : price increases
D : price decreases



b) Find the transition matrix.

$$P = \begin{array}{c c} & \begin{array}{cc} I & D \end{array} \\ \begin{array}{c} I \\ D \end{array} & \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix} \end{array} \leftarrow \text{next day}$$

today

c) There is an 80% probability that the stock's price will decrease today. Find the probability that the price increases two days from now.

$$S_0 = \begin{array}{c} \begin{array}{cc} I & D \end{array} \\ \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \end{array} \quad \text{today}$$

$$S_1 = S_0 P$$
$$= \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix}$$

Example Continued...

$$= \begin{bmatrix} 0.57 & 0.43 \end{bmatrix}$$ tomorrow

$$S_2 = S_1 P$$

$$= \begin{bmatrix} 0.57 & 0.43 \end{bmatrix} \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix}$$

$$\begin{array}{cc} I & D \end{array}$$

$$= \begin{bmatrix} 0.607 & 0.393 \end{bmatrix}$$ 2 days from now

The probability that the price increases two days from now is 60.7%

Comment: Depending on the problem, the next state matrix could represent the next purchase, the next day, the next week, the next year etc.