6.2 General Systems of Equations

In this section we'll look at systems of equations with no solution or with infinitely-many solutions.

Example: Solve using Gauss-Jordan Elimination:

$$\begin{aligned} x + 2y + z &= 9 \\ x + 3y + 3z &= 12 \\ x + 4y + 5z &= 1 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & \# \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & -8 \end{array}\right]$$

$$R_3 - 2R_2 \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & -14 \end{array}\right]$$

$0x + 0y + 0z = -14$ is impossible.

The system of equations has no solution.

Fact: If you see a row like $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \text{nonzero} \end{array}\right]$, then the system has no solution.

Example: Solve using Gauss-Jordan Elimination:

$$\begin{aligned} x + 2y + \ z &= 9 \\ x + 3y + 3z &= 12 \\ x + 4y + 5z &= 15 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & \# \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 15 \end{array}\right]} \end{array}$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 2R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Cannot make this 1.
Row operations are done.

Circle the leading 1 in each row:

$$\begin{array}{cccc} x & y & z & \# \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]} \end{array}$$

If a variable has no circle in its column then it is a "free variable"

$x - 3z = 3 \Rightarrow$

$y + 2z = 3 \Rightarrow$

$$\boxed{\begin{aligned} &z = \text{any value} \\ &x = 3 + 3z \\ &y = 3 - 2z \end{aligned}}$$

Example: Find three solutions to the system on the previous page.

The system has infinitely-many solutions.

$z=0 \implies x=3,\ y=3$

$(x,y,z)=(3,3,0)$

$z=1 \implies x=6,\ y=1$

$(x,y,z)=(6,1,1)$

$z=2 \implies x=9,\ y=-1$

$(x,y,z)=(9,-1,2)$

etc.

Example: Solve the following systems. Assume the variables are x and y, in that order.

a) $$\begin{array}{c} \begin{array}{ccc} x & y & \# \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \leftarrow x = 4 \\ \leftarrow y = 5 \\ \leftarrow \text{no info} \end{array}$$

$$(x, y) = (4, 5)$$

b) $$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array}\right]$$

The system has no solution.

Example: A store sells three items priced at \$7, \$10 and \$13. Amanda wants to buy 15 items in total and she wants to spend exactly \$150. Give three options for what Amanda could buy.

Let x = # of \$7 items Amanda buys
" y " \$10 "
" z " \$13 "

# of items: $x + y + z = 15$

\$: $7x + 10y + 13z = 150$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 7 & 10 & 13 & 150 \end{array}\right]$$

$R_2 - 7R_1$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 0 & 3 & 6 & 45 \end{array}\right]$$

$\frac{R_2}{3}$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 0 & 1 & 2 & 15 \end{array}\right]$$

$R_1 - R_2$ $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 15 \end{array}\right]$$

As close as possible to diagonal form..

$\rightarrow$

Example Continued...

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 15 \end{array}\right]$$

$x - z = 0 \Rightarrow$ $x = z$

$y + 2z = 15 \Rightarrow$ $y = 15 - 2z$

$$\boxed{\begin{array}{l} z = \text{any value} \\ x = z \\ y = 15 - 2z \end{array}}$$

Option #1 : $z = 0 \Rightarrow x = 0, \quad y = 15$

$(x, y, z) = (0, 15, 0)$

Option # 2 : $z = 1 \Rightarrow x = 1, \quad y = 13$

$(x, y, z) = (1, 13, 1)$

Option #3 : $z = 2 \Rightarrow x = 2, \quad y = 11$

$(x, y, z) = (2, 11, 2)$

etc.