6.1 Systems of Equations with Unique Solutions

In this chapter we'll focus on solving systems of equations. This will prepare us for more applied problems in Chapter 7.

Definition: A **matrix** is a rectangular array of numbers. We say "one matrix" and "two or more matrices".

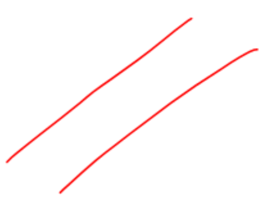
Example: Write the following system of equations in matrix form:

$$\begin{aligned} 9x - 2y &= 18 \\ 3x + 4y &= 27 \end{aligned}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & \# \end{array} \\ \left[\begin{array}{cc|c} 9 & -2 & 18 \\ 3 & 4 & 27 \end{array}\right] \end{array}$$

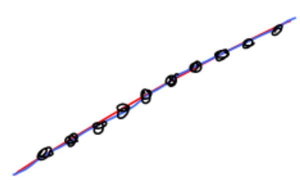
Fact: A system of equations can have no solution, one unique solution, or infinitely-many solutions.

Example: Draw a picture to illustrate each of the three scenarios in the fact above.



No solution

one unique solution



infinitely-many solutions

Definition: A matrix is in **diagonal form** if it looks like $\left[\begin{array}{cc|c} 1 & 0 & \# \\ 0 & 1 & \# \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right]$, where # represents any number.

Fact: Row Operations
To solve a system of equations, we perform operations on the matrix form until it becomes diagonal form (or as close as possible to diagonal form). There are three types of operations we are allowed to perform:

1) Swap any two rows

2) Multiply or divide any row by a nonzero number

3) (Current Row) - #(Pivot Row)

Comment: These operations do not change the solution to the system of equations.

Example: Solve:

$$\begin{aligned} 3x + 6y &= 30 \\ 2x + 8y &= 32 \end{aligned}$$

$$\begin{array}{ccc} x & y & \# \\ \end{array}$$
$$\left[\begin{array}{cc|c} 3 & 6 & 30 \\ 2 & 8 & 32 \end{array}\right]$$

Get a 1 in the top left, "the pivot"

$$\frac{R_1}{3} \quad \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 2 & 8 & 32 \end{array}\right]$$

Get 0's in the rest of Column 1 by:
(Current Row) − # (Pivot Row)

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 4 & 12 \end{array}\right]$$

Get a 1 in next diagonal position

$$\frac{R_2}{4} \quad \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 1 & 3 \end{array}\right]$$

→

Example Continued...

$\boxed{\begin{array}{l}\text{Get 0's in the rest of Column 2 by:}\\ \quad\text{(Current Row)} - \#\,\text{(Pivot Row)}\end{array}}$

$$R_1 - 2R_2 \quad \begin{array}{ccc} x & y & \# \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right]$$

diagonal form

$$1x + 0y = 4 \Rightarrow x = 4$$
$$0x + 1y = 3 \Rightarrow y = 3$$

$$(x, y) = (4, 3)$$

Comment: This process is called **Gauss-Jordan Elimination**.

Example: Solve using Gauss-Jordan Elimination:

$$\begin{aligned} 4x - 10y + 10z &= -18 \\ x - \ 3y + \ 4z &= -8 \\ -3x + \ 11y - \ 5z &= 25 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & \# \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 4 & -10 & 10 & -18 \\ 1 & -3 & 4 & -8 \\ -3 & 11 & -5 & 25 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -8 \\ 4 & -10 & 10 & -18 \\ -3 & 11 & -5 & 25 \end{array}\right]$$

$R_2 - 4R_1$

$R_3 + 3R_1$
$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -8 \\ 0 & 2 & -6 & 14 \\ 0 & 2 & 7 & 1 \end{array}\right]$$

(Current Row) − # (Pivot Row)

$\frac{R_2}{2}$
$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -8 \\ 0 & 1 & -3 & 7 \\ 0 & 2 & 7 & 1 \end{array}\right]$$

$R_1 + 3R_2$

$R_3 - 2R_2$
$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 13 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 13 & -13 \end{array}\right]$$

$\longrightarrow$

Example Continued...

$$\frac{R_3}{13} \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 13 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & \# \end{array} \\ \begin{array}{c} R_1+5R_3 \\ R_2+3R_3 \\ \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array}$$

$$x = 8$$
$$y = 4$$
$$z = -1$$

$$(x, y, z) = (8, 4, -1)$$

Example: Each hat takes 3 hours and \$2 to produce. Each coat takes 6 hours and \$8 to produce. You must spend exactly 30 hours and \$32. How many hats and coats can you produce?

Let x = # of hats you can produce
y = " coats "

Hours: $3x + 6y = 30$

\$: $2x + 8y = 32$

See pages 3 and 4 of this section:

$(x, y) = (4, 3)$