5.1 Expected Value

In this chapter we'll look at two specific probability concepts that come up in business, science, and social science.

Notation: In Chapter 5 we'll write probabilities as P(E) instead of Pr(E). Feel free to use either notation.

Notation: In Chapter 5 we'll write combinations in the format $5C3$ instead of $C(5,3)$. Feel free to use either notation.

(Sugg. HW problems are taken from various textbooks)

Definition: A **random variable** assigns a number to each outcome of an experiment. Random variables are written X.

random variable

Example: Three fair coins are flipped. Let X be the number of heads that appear. Find the probability distribution of X.

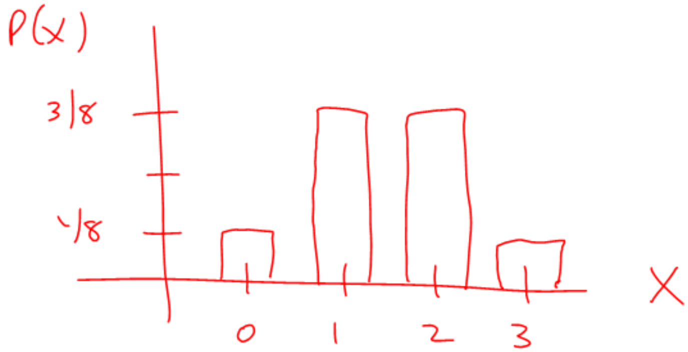
X	Description	# of Ways	P(x)
0	TTT	1	1/8
1	HTT, THT, TTH	3	3/8
2	HHT, HTH, THH	3	3/8
3	HHH	1	1/8

total = 8

Probability distribution of X:

X	P(x)
0	1/8
1	3/8
2	3/8
3	1/8

Example: Referring to the previous example, let's draw the histogram (bar chart) that represents X.



Definition: The **expected value** of a random variable X is the theoretical average of X if the experiment were repeated infinitely-many times. It is sometimes called the mean of X.

Notation: The expected value of X is written μ or $E(X)$. Feel free to use either notation.

Fact: $\mu = x_1p_1 + x_2p_2 + \ldots + x_np_n$ where:

$x_1, x_2, \ldots, x_n$ are the values of X

$p_1, p_2, \ldots, p_n$ are their respective probabilities

Example: A fair die is rolled. Let X be the number rolled. Find the expected value of X.

X	P(x)
1	1/6
2	1/6
3	
4	↓
5	
6	1/6

$\mu = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + \ldots + 6\left(\frac{1}{6}\right)$

$\mu = 3.5$

or $E(X) = 3.5$

Example: A box contains nine \$5 bills and six \$10 bills. You pay \$8 and randomly draw a bill from the box. Let X be your net winnings (in dollars). Find the expected value of X.

X = net winnings
= amount won − cost

\$5	\$10
9	6

	X	$P(X)$
draw \$5 →	-3	$9/15$
draw \$10 →	2	$6/15$

$$\mu = -3\left(\frac{9}{15}\right) + 2\left(\frac{6}{15}\right)$$

$$\mu = -1$$

or $E(X) = -1$

We expect to lose \$1, on average, each time we play.

Example: You insure a used car worth $4,000 against theft for one year by paying a premium of $112. The probability that the car is stolen during this year is 1.3%. Find your expected net gain (in dollars) on the insurance policy.

X = net gain

= insurance payout − premium

	X	$P(x)$	
car is stolen →	$4000-112 = 3888$	0.013	
car isn't stolen →	-112	0.987	$\leftarrow 1-0.013$

$$\mu = 3888(0.013) + (-112)(0.987)$$
$$= -60$$

or $E(X) = -60$

Expect to lose $60, on average, each year.
(But we're protected against a large loss.)

Example: A shipment contains ten good and five defective items. We randomly select three items from the shipment. Find the expected number of good items that are selected.

X = # of good items selected

Let G: good D: defective

$\boxed{10G \mid 5D}$

X	Description	# of ways	P(X)
0	3D	$5C3 = 10$	$10/455$
1	1G and 2D	$10C1 \times 5C2 = 100$	$100/455$
2	2G and 1D	$10C2 \times 5C1 = 225$	$225/455$
3	3G	$10C3 = 120$	$120/455$

total = 455

$$\mu = 0\left(\frac{10}{455}\right) + 1\left(\frac{100}{455}\right) + 2\left(\frac{225}{455}\right) + 3\left(\frac{120}{455}\right)$$

$$= 2$$

or $E(X) = 2$

We expect to select 2 good items, on average.

Definition: A game is **fair** if the expected net winnings is equal to zero.

Example: You pay \$1 to roll a fair die. If you roll a 1 or a 6, you win \$5. Otherwise, you must pay k more dollars. Find k so that the game is fair.

X = net winnings = amount won − cost

	X	$P(x)$
roll 1 or 6 →	4	$\frac{2}{6}$
roll 2,3,4 or 5 →	$-1-k$	$\frac{4}{6}$

Game is fair

$\Rightarrow \mu = 0$

$$4\left(\frac{2}{6}\right) + (-1-k)\left(\frac{4}{6}\right) = 0$$

Multiply by 6:

$$4(2) + (-1-k)(4) = 0$$
$$8 - 4 - 4k = 0$$
$$4 = 4k$$
$$1 = k$$