

3.5 Permutations and Combinations

Definition: A **permutation** is an ordered selection of r objects from a group of n objects.

Notation: The number of permutations is written $P(n, r)$.

Fact: $P(n, r) = n \times (n - 1) \times (n - 2) \times \cdots \times (n - r + 1)$

r factors

Example: Compute the following in two ways: using the formula above and using your calculator.

a) $P(40, 3)$

$$\begin{aligned} &= 40 \times 39 \times 38 \\ &= 59,280 \end{aligned}$$

$$\boxed{40} \boxed{2ndF} \boxed{nPr} \boxed{3} \boxed{=}$$

b) $P(4, 4)$

$$\begin{aligned} &= 4 \times 3 \times 2 \times 1 \\ &= 24 \end{aligned}$$

c) $P(8, 1)$

$$= 8$$

Example: How many two-letter “words” (including nonsense words) can be formed from A,B,C,D if repetition is not allowed?

$$\begin{aligned} &P(4, 2) \\ &= 4 \times 3 \\ &= 12 \end{aligned}$$

AB	AC	AD	BC	BD	CD
BA	CA	DA	CB	DB	DC

Notation: $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$
 It is pronounced “n factorial.”

Example: Compute the following in two ways: using the formula above and using your calculator.

a) $5!$

$$= 5 \times 4 \times 3 \times 2 \times 1 \\ = 120$$

$$5 \boxed{\text{2nd F}} \boxed{n!} \boxed{=}$$

b) $3!$

$$= 3 \times 2 \times 1 \\ = 6$$

c) $1!$

$$= 1$$

Comment: Note that $0! = 1$ by definition. You can confirm this on your calculator.

Example: How many ways are there to arrange four books in a row?

$$4! \quad \text{or} \quad P(4, 4) \quad \text{or} \quad 4 \times 3 \times 2 \times 1 \\ = 24$$

Fact: $P(k, k) = k!$ for $k = 0, 1, 2, \dots$

$$P(6, 6) = 6! \\ P(3, 3) = 3! \\ \text{etc.}$$

Definition: A **combination** is an unordered selection of r objects from a group of n objects.

Notation: The number of combinations is written $C(n, r)$.

Fact: $C(n, r) = \frac{P(n, r)}{r!}$

Example: Compute the following using your calculator.

a) $C(40, 3)$

$= 9,880$

$\boxed{40} \boxed{2nd F} \boxed{nCr} \boxed{3} \boxed{=}$

b) $C(4, 4)$

$= 1$

c) $C(8, 1)$

$= 8$

Example: How many ways are there to select two people from a group of four?

$C(4, 2) = 6$

$\{A, B\}$	$\{A, C\}$	$\{A, D\}$
$\{B, C\}$	$\{B, D\}$	$\{C, D\}$

Example: We have interviewed 20 candidates for a job. How many ways are there to select our 1st, 2nd and 3rd choice?

ordered

$$P(20, 3) \text{ or } 20 \times 19 \times 18$$
$$= 6,840$$

Example: A class has 45 students. How many ways are there to form a four-person team?

unordered

$$C(45, 4)$$
$$= 148,995$$

Example: In a batch of 150 numbered phones, four are defective.

a) How many ways are there to select three phones from the batch?

unordered

$$C(150, 3)$$
$$= 551,300$$

b) How many ways are there to select three defective phones from the batch?

unordered

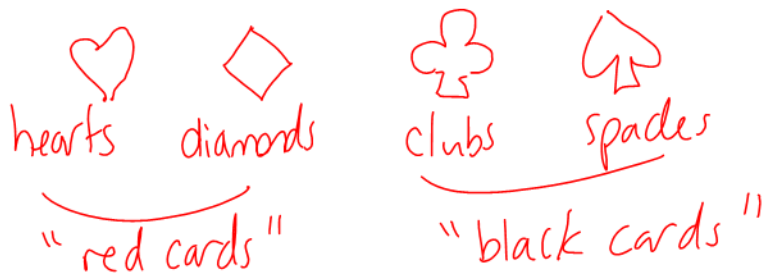
Select 3 of 4 defective phones

$$C(4, 3)$$
$$= 4$$

The standard deck of cards sometimes comes up in counting problems.

Example: In this example we'll write down everything we need to know about the standard deck of cards.

52 cards divided into 4 suits :



Each suit has 13 cards: $A, 2, 3, \dots, 10, J, Q, K$

Example: How many five-card hands from a standard deck have:

a) only hearts?

unordered
Select 5 of 13 hearts
 $C(13, 5) = 1287$

b) no hearts?

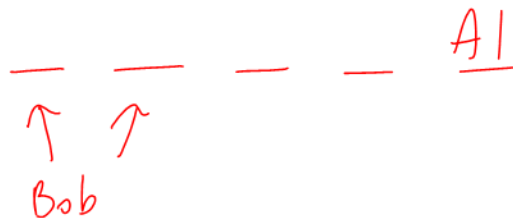
unordered
 $52 - 13 = 39$ non-hearts in the deck
Select 5 of 39 non-hearts
 $C(39, 5) = 575, 757$

Example: How many ways are there to select four of seven books and arrange them in a row?

$$\text{Method 1: } P(7,4) = 7 \times 6 \times 5 \times 4 = 840$$

$$\begin{aligned} \text{Method 2: } & C(7,4) \times 4! \\ & \begin{array}{cc} \text{Select the} & \text{Order the} \\ \text{4 books} & \text{4 books} \end{array} \\ & = 35 \times 24 \\ & = 840 \end{aligned}$$

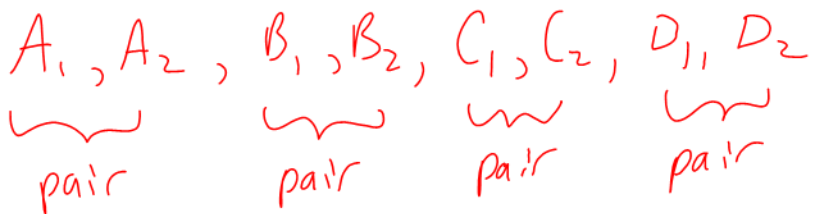
Example: Five students are giving presentations. How many orders are possible if Al goes last and Bob goes first or second?



$$\boxed{2} \times \boxed{3!} = 2 \times 6 = 12$$

Select Bob's position Order the other 3 people

Example: How many ways are there to arrange four pairs of people in a row so that each pair is adjacent?

Call the people $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$


e.g. $B_2 B_1 D_1 D_2 A_1 A_2 C_2 C_1$

$$\begin{array}{ccccccccc}
 \boxed{4!} & \times & \boxed{2} & \times & \boxed{2} & \times & \boxed{2} & \times & \boxed{2} \\
 \text{Order} & & \text{Order} & & \text{Order} & & \text{Order} & & \text{Order} \\
 \text{the 4 pairs} & & A_1, A_2 & & B_1, B_2 & & C_1, C_2 & & D_1, D_2
 \end{array}$$

$$\begin{aligned}
 &= 24 \times 2^4 \\
 &= 384
 \end{aligned}$$