## 3.5 Permutations and Combinations

**Definition:** A **permutation** is an ordered selection of $r$ objects from a group of $n$ objects.

**Notation:** The number of permutations is written $P(n, r)$.

**Fact:** $P(n,r) = \underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r \text{ factors}}$

**Example:** Compute the following in two ways: using the formula above and using your calculator.

a) $P(40, 3)$

$= 40 \times 39 \times 38$

$= 59{,}280$

$\boxed{40}\ \boxed{2ndF}\ \boxed{nPr}\ \boxed{3}\ \boxed{=}$

b) $P(4, 4)$

$= 4 \times 3 \times 2 \times 1$

$= 24$

c) $P(8, 1)$

$= 8$

**Example:** How many two-letter "words" (including nonsense words) can be formed from A,B,C,D if repetition is not allowed?

$P(4,2)$

$= 4 \times 3$

$= 12$

| | | | | | |
|---|---|---|---|---|---|
| AB | AC | AD | BC | BD | CD |
| BA | CA | DA | CB | DB | DC |

**Notation:** $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
It is pronounced "n factorial."

**Example:** Compute the following in two ways: using the formula above and using your calculator.

a) 5!

$= 5 \times 4 \times 3 \times 2 \times 1$
$= 120$

5 $\boxed{\text{2nd F}}$ $\boxed{n!}$ $\boxed{=}$

b) 3!

$= 3 \times 2 \times 1$
$= 6$

c) 1!

$= 1$

**Comment:** Note that $0! = 1$ by definition. You can confirm this on your calculator.

**Example:** How many ways are there to arrange four books in a row?

$4!$ or $P(4,4)$ or $4 \times 3 \times 2 \times 1$

$= 24$

**Fact:** $P(k,k) = k!$ for $k = 0, 1, 2, ...$

$P(6,6) = 6!$
$P(3,3) = 3!$
etc.

**Definition:** A **combination** is an unordered selection of $r$ objects from a group of $n$ objects.

**Notation:** The number of combinations is written $C(n, r)$.

**Fact:** $C(n, r) = \frac{P(n,r)}{r!}$

**Example:** Compute the following using your calculator.

a) $C(40, 3)$

$= 9,880$

$\boxed{40}\ \boxed{2nd\ F}\ \boxed{nCr}\ \boxed{3}\ \boxed{=}$

b) $C(4, 4)$

$= 1$

c) $C(8, 1)$

$= 8$

**Example:** How many ways are there to select two people from a group of four?

$C(4,2) = 6$

$\{A,B\}$ $\{A,C\}$ $\{A,D\}$

$\{B,C\}$ $\{B,D\}$ $\{C,D\}$

**Example:** We have interviewed 20 candidates for a job. How many ways are there to select our 1st, 2nd and 3rd choice?

ordered

$P(20,3)$ or $20 \times 19 \times 18$

$= 6{,}840$

**Example:** A class has 45 students. How many ways are there to form a four-person team?

unordered

$C(45,4)$

$= 148{,}995$

**Example:** In a batch of 150 numbered phones, four are defective.

a) How many ways are there to select three phones from the batch?

unordered

$C(150,3)$

$= 551{,}300$

b) How many ways are there to select three defective phones from the batch?

unordered

Select 3 of 4 defective phones

$C(4,3)$

$= 4$

The standard deck of cards sometimes comes up in counting problems.

**Example:** In this example we'll write down everything we need to know about the standard deck of cards.

52 cards divided into 4 suits:

♡ ♢ ♧ ♤

$\underbrace{\text{hearts} \quad \text{diamonds}}_{\text{"red cards"}} \qquad \underbrace{\text{clubs} \quad \text{spades}}_{\text{"black cards"}}$

Each suit has 13 cards: A, 2, 3, ..., 10, J, Q, K

**Example:** How many five-card hands from a standard deck have:

a) only hearts?

unordered

Select 5 of 13 hearts

$C(13,5) = 1287$

b) no hearts?

unordered

$52-13=39$ non-hearts in the deck

Select 5 of 39 non-hearts

$C(39,5) = 575,757$

**Example:** How many ways are there to select four of seven books and arrange them in a row?

Method 1 : $P(7,4) = 7 \times 6 \times 5 \times 4$
$= 840$

Method 2 : $\underbrace{C(7,4)}_{\text{Select the 4 books}} \times \underbrace{4!}_{\text{Order the 4 books}}$

$= 35 \times 24$

$= 840$

**Example:** Five students are giving presentations. How many orders are possible if Al goes last and Bob goes first or second?

__ __ __ __ Al

↑ ↑
Bob

$\underbrace{\boxed{2}}_{\text{Select Bob's position}} \times \underbrace{\boxed{3!}}_{\text{Order the other 3 people}} = 2 \times 6 = 12$

**Example:** How many ways are there to arrange four pairs of people in a row so that each pair is adjacent?

Call the people $\underbrace{A_1, A_2}_{\text{pair}}, \underbrace{B_1, B_2}_{\text{pair}}, \underbrace{C_1, C_2}_{\text{pair}}, \underbrace{D_1, D_2}_{\text{pair}}$

e.g. $B_2 B_1 D_1 D_2 A_1 A_2 C_2 C_1$

$$\underbrace{\boxed{4!}}_{\text{Order the 4 pairs}} \times \underbrace{\boxed{2}}_{\text{Order } A_1, A_2} \times \underbrace{\boxed{2}}_{\text{Order } B_1, B_2} \times \underbrace{\boxed{2}}_{\text{Order } C_1, C_2} \times \underbrace{\boxed{2}}_{\text{Order } D_1, D_2}$$

$= 24 \times 2^4$

$= 384$