Test Average: 81%

Test 3 Tues March 28th

Final Exam Mon April 17th 1:30pm CBA 285

4.7 Cont'd

Recap

Cauchy-Euler DE:

$$x^3y''' + 3x^2y'' + 3xy' - 3y = 0$$

Auxiliary Equation for Cauchy-Euler DE:

$$x^3y''' \to m(m-1)(m-2)$$

$$x^2y'' \to m(m-1)$$

$$xy' \to m$$

$$y \to 1$$

Roots	Solutions to Cauchy-Euler DE
$m_1, m_2, \ldots$	$x^{m_1}, x^{m_2}, \ldots$
$m_1, m_1, m_1, \ldots$	$x^{m_1}, x^{m_1}\ln x, x^{m_1}(\ln x)^2, \ldots$
$\alpha \pm \beta i$	$x^{\alpha}[C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)]$

Ex: Solve $x^3y''' + 3x^2y'' + 3xy' - 3y = 0$

Cauchy-Euler DE

$$m(m-1)(m-2) + 3m(m-1) + 3m - 3 = 0$$

$$m(m^2-3m+2) + 3m^2 \underbrace{-3m + 3m} - 3 = 0$$

$$m^3 - 3m^2 + 2m + 3m^2 \qquad -3 = 0$$

$$m^3 + 2m - 3 = 0$$

Rational Roots Theorem

$p: \pm 1, \pm 3$

$q: \pm 1$

(Possibilities for m) $\frac{p}{q}: \pm 1, \pm 3$

Check $m = -1$: $(-1)^3 + 2(-1) - 3 = 0$? No

$m = 1$: $1^3 + 2(1) - 3 = 0$? YES

$m = 1$ is a root of $m^3 + 2m - 3$

$\Rightarrow$ $m - 1$ " factor "

$$
\begin{array}{r|l}
 & m^2 + m + 3 \\
\hline
(m-1) & m^3 + 0m^2 + 2m - 3 \\
 & -(m^3 - m^2) \\
\hline
 & m^2 + 2m - 3 \\
 & -(m^2 - m) \\
\hline
 & 3m - 3 \\
 & -(3m - 3) \\
\hline
 & 0
\end{array}
$$

$$(m-1)(m^2 + m + 3) = 0$$

$m = 1$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{1 - 4(1)(3)}}{2}$$

$$m = \frac{-1 \pm \cancel{\sqrt{-11}}}{2} \quad \sqrt{11}\, i$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i \qquad \left(\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{11}}{2}\right)$$

$$y = x^{-1/2}\left[C_1 \cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}\ln x\right)\right] + C_3 x$$

Ex: Solve $x^2 y'' - 6xy' + 12y = \frac{1}{x}$

Variation of Parameters

1) y_c

Cauchy-Euler DE

$$m(m-1) - 6m + 12 = 0$$
$$m^2 - m - 6m + 12 = 0$$
$$m^2 - 7m + 12 = 0$$
$$(m-3)(m-4) = 0$$
$$m = 3, 4$$
$$y_1 = x^3 \qquad y_2 = x^4$$
$$y_c = C_1 x^3 + C_2 x^4$$

2) W, W_1, W_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^4 \\ x^{-3} & \sim \end{vmatrix} = -x$$

(!) $\boxed{\text{Standard Form} \quad y'' + \dots = x^{-3} \leftarrow f(x)}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & 0 \\ \sim & x^{-3} \end{vmatrix}$$

$$= 1$$

3) u_1' and u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-x}{x^6}$$

$$= -x^{-5}$$

$$u_1 = \int -x^{-5}\,dx$$

$$= \frac{1}{4}x^{-4}$$ Don't use a constant.

4) u_2' and u_2

$$u_2' = \frac{W_2}{W}$$

$$= x^{-6}$$

$$u_2 = \int x^{-6}\,dx$$

$$= -\frac{1}{5}x^{-5}$$ Don't use a constant.

5) y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}x^{-4}(x^3) - \frac{1}{5}x^{-5}(x^4)$$ ✓

$$= \frac{1}{4}x^{-1} - \frac{1}{5}x^{-1}$$

$$= \frac{1}{20}x^{-1}$$

6) y

$$y = y_c + y_p$$

$$y = C_1 x^3 + C_2 x^4 + \frac{1}{20} x^{-1}$$

7) ~~Initial Conditions~~