4.6 Cont'd

Ex: Solve $3y'' + 15y' + 18y = 3\csc e^{x}$

Standard Form $y'' + 5y' + 6y = \underbrace{\csc e^{x}}_{f(x)}$

1) y_c

$$m^2 + 5m + 6 = 0$$
$$(m+2)(m+3) = 0$$
$$m = -2, -3$$

$$y_1 = e^{-2x} \qquad y_2 = e^{-3x}$$

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

2) W, W_1, W_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

$$= -3e^{-5x} + 2e^{-5x}$$

$$= -e^{-5x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & e^{-3x} \\ \csc e^{x} & \sim \end{vmatrix}$$ Use Standard Form for $f(x)$

$$= -e^{-3x} \cos e^{x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{-2x} & 0 \\ \sim & \cos e^{x} \end{vmatrix}$$

Use Standard Form for $f(x)$

$$= e^{-2x} \cos e^{x}$$

3) u_1' and u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-e^{-3x} \cos e^{x}}{-e^{-5x}}$$

$$= e^{2x} \cos e^{x}$$

$$u_1 = \int e^{2x} \cos e^{x} \, dx$$

$$\boxed{\begin{aligned} u &= e^{x} \\ du &= e^{x} \, dx \end{aligned}}$$

$$= \int e^{x} \cos e^{x} \left(e^{x} dx\right)$$

$$= \int u \cos u \, du$$

	D	I
$+$	u	$\cos u$
$-$	1	$\sin u$
		$-\cos u$

$$= u\sin u + \cos u$$

Don't use a constant

$$= e^{x}\sin e^{x} + \cos e^{x}$$

4) u_2' and u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{e^{-2x}\cos e^{x}}{-e^{-5x}}$$

$$= -e^{3x}\cos e^{x}$$

$$u_2 = \int -e^{3x}\cos e^{x}\,dx$$

$$\boxed{\begin{aligned} u &= e^{x} \\ du &= e^{x}dx \end{aligned}}$$

$$= \int \underbrace{-e^{2x}}_{-(e^{x})^2}\cos e^{x}\,(e^{x}dx)$$

$$= \int -u^2\cos u\,du$$

	D	I
$+$	$-u^2$	$\cos u$
$-$	$-2u$	$\sin u$
$+$	-2	$-\cos u$
		$-\sin u$

$$= -u^2\sin u - 2u\cos u + 2\sin u$$

Don't use a constant

$$= -e^{2x}\sin e^{x} - 2e^{x}\cos e^{x} + 2\sin e^{x}$$

5) y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(e^{x} \sin e^{x} + \cos e^{x}\right) e^{-2x}$$

$$+ \left(-e^{2x} \sin e^{x} - 2e^{x} \cos e^{x} + 2\sin e^{x}\right) e^{-3x} \checkmark$$

or $\quad -e^{-2x} \cos e^{x} + 2e^{-3x} \sin e^{x}$ ✓

6) General Solution y

$$y = y_c + y_p$$

7) ~~Initial Conditions~~

Ex: Solve $y'' - 3y' + 2y = x^3$ using Variation of Parameters.

Note: Could also use Section 4.4 "Undetermined Coefficients"

1) y_c

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$y_1 = e^{x} \qquad y_2 = e^{2x} \qquad y_c = C_1 e^{x} + C_2 e^{2x}$

2) W, W_1, W_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}$$

$$= 2e^{3x} - e^{3x}$$

$$= e^{3x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & e^{2x} \\ x^3 & \sim \end{vmatrix}$$

Use $\underline{\text{standard form}}$ for $f(x)$

$$= -x^3 e^{2x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{x} & 0 \\ \sim & x^3 \end{vmatrix}$$

$$= x^3 e^{x}$$

3) u_1' and u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-x^3 e^{2x}}{e^{3x}}$$

$$= -x^3 e^{-x}$$

$$u_1 = \int -x^3 e^{-x} \, dx$$

	D	I
+	$-x^3$	e^{-x}
−	$-3x^2$	$-e^{-x}$
+	$-6x$	e^{-x}
−	-6	$-e^{-x}$
		e^{-x}

$$= (x^3 + 3x^2 + 6x + 6)e^{-x}$$

No constant

4) u_2' and u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{x^3 e^{x}}{e^{3x}}$$

$$= x^3 e^{-2x}$$

$$u_2 = \int x^3 e^{-2x} \, dx$$

	D	I
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2} e^{-2x}$
+	$6x$	$\frac{1}{4} e^{-2x}$
−	6	$-\frac{1}{8} e^{-2x}$
		$\frac{1}{16} e^{-2x}$

$$u_2 = \left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{6x}{8} - \frac{6}{16}\right)e^{-2x}$$

No constant

5) y_p

$$\boxed{y_p = u_1 y_1 + u_2 y_2}$$ ✓

or $y_p = \frac{x^3}{2} + \frac{9x^2}{4} + \frac{21x}{4} + \frac{45}{8}$ ✓

6) y

$$\boxed{y = y_c + y_p}$$ ✓

~~7) Initial Conditions~~