<u>Note</u>: The method from Section 4.4 is called "Undetermined Coefficients"

4.6 Variation of Parameters

Find a particular solution y_p for $y'' + ay' + by = f(x)$ when the list $f(x), f'(x), f''(x), \ldots$ contains infinitely-many like terms.

Idea: Find solutions y_1 and y_2 to $y'' + ay' + by = 0$

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $\quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

Integrate to get u_1 and u_2

Then $y_p = u_1 y_1 + u_2 y_2$

<u>Ex</u>: Solve $5y'' + 20y = 5\csc 2x$

Standard Form $y'' + 4y = \underbrace{\csc 2x}_{f(x)}$

1) y_c

$$y'' + 4y = 0$$
$$m^2 + 4 = 0$$
$$m^2 = -4$$
$$m = \pm \cancel{\sqrt{-4}} \; \sqrt{4}\sqrt{-1}$$
$$m = \pm 2i \qquad (\alpha = 0, \beta = 2)$$

$$y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$
$$y_c = C_1 \cos 2x + C_2 \sin 2x$$
$$y_1 = \cos 2x \qquad y_2 = \sin 2x$$

2) W, W_1, W_2

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$
$$= 2\cos^2 2x + 2\sin^2 2x$$
$$= 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$ Use standard form for $f(x)$

$$= \begin{vmatrix} 0 & \sin 2x \\ \csc 2x & \sim \end{vmatrix}$$
$$= -\sin 2x \csc 2x$$
$$= -1$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & 0 \\ \sim & \csc 2x \end{vmatrix}$$

Use standard form for $f(x)$

$$= \cos 2x \csc 2x$$

$$= \frac{\cos 2x}{\sin 2x}$$

$$= \cot 2x$$

3) u_1' and u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-1}{2}$$

$$u_1 = \int -\frac{1}{2}\,dx$$

$$u_1 = -\frac{x}{2}$$

Don't use a constant.

4) u_2' and u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{2}\cot 2x$$

$$u_2 = \int \frac{1}{2}\cot 2x\,dx$$

$$= \frac{1}{4}\ln|\sin 2x|$$

Don't use a constant.

$$\boxed{\begin{aligned} &\int \cot x\,dx = \ln|\sin x| + C \\ &\int \cot kx\,dx = \frac{1}{k}\ln|\sin kx| + C \end{aligned}}$$

5) $y_p = u_1 y_1 + u_2 y_2$

$$y_p = -\frac{x}{2}\cos 2x + \frac{1}{4}\ln|\sin 2x| \sin 2x$$

6) General Solution

$$y = y_c + y_p$$

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{x}{2}\cos 2x + \frac{1}{4}\ln|\sin 2x| \sin 2x$$

~~7) Initial Conditions~~