4.3 Cont'd

Roots of Auxiliary Equation	Solutions to DE
$m_1, m_2, m_3, \dots$	$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \dots$
$m_1, m_1, m_1, \dots$	$e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \dots$
$\alpha \pm \beta i$	$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x$

<u>Ex</u>: a) If $m = 1, 3, 3$

$y_1 = e^{x}$ $\quad y_2 = e^{3x}$ $\quad y_3 = xe^{3x}$

b) If $m = 4, 4, 3 \pm 5i$

$y_1 = e^{4x}$ $\quad y_2 = xe^{4x}$ $\quad y_3 = e^{3x}\cos 5x$ $\quad y_4 = e^{3x}\sin 5x$

General Solution

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$$

$$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 e^{3x}\cos 5x + C_4 e^{3x}\sin 5x$$

$$y = (C_1 + C_2 x)e^{4x} + e^{3x}(C_3\cos 5x + C_4 \sin 5x)$$

Rational Roots Theorem

Any rational roots of a polynomial are of the form $\frac{p}{q}$, where p divides the constant and q divides the leading coefficient.

Ex: Solve $2y''' - 7y'' + 7y' - 2y = 0$ $\quad \begin{cases} y(0) = 3 \\ y'(0) = 5 \\ y''(0) = 9 \end{cases}$

$$2m^3 - 7m^2 + 7m - 2 = 0$$

p divides -2 $\qquad p: \pm 1, \pm 2$

q divides 2 $\qquad q: \pm 1, \pm 2$

$$\frac{p}{q}: \underbrace{\pm 1, \pm \tfrac{1}{2}, \pm 2}_{\text{possible roots}}$$

$m = -1$: $\quad 2(-1)^3 - 7(-1)^2 + 7(-1) - 2 = 0\,?$ $\quad$ No

$m = 1$: $\quad 2(1)^3 - 7(1)^2 + 7(1) - 2 = 0$ $\quad \checkmark$

$m = 1$ is a root $\Rightarrow$ $(m-1)$ is a factor

$$\begin{array}{r|l}
 & 2m^2 - 5m + 2 \\ \hline
(m-1) & 2m^3 - 7m^2 + 7m - 2 \\
 & -(2m^3 - 2m^2) \\ \hline
 & -5m^2 + 7m - 2 \\
 & -(-5m^2 + 5m) \\ \hline
 & 2m - 2 \\
 & -(2m - 2) \\ \hline
 & 0
\end{array}$$

Auxiliary Equation: $(m-1)(2m^2 - 5m + 2) = 0$

$$m = 1 \qquad\qquad m = \frac{5 \pm \sqrt{25 - 4(2)(2)}}{4}$$

$$m = \frac{5 \pm \cancel{\sqrt{9}}\ 3}{4}$$

$$m = 2, \quad \frac{1}{2}$$

$$y_1 = e^{x} \qquad y_2 = e^{2x} \qquad y_3 = e^{\frac{x}{2}}$$

General Solution $\boxed{y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{\frac{x}{2}}}$

$y=3$, $x=0$:
$$3 = C_1 + C_2 + C_3 \qquad (1)$$

$$y' = C_1 e^{x} + 2C_2 e^{2x} + \frac{C_3}{2} e^{\frac{x}{2}}$$

$y'=5$, $x=0$:
$$5 = C_1 + 2C_2 + \frac{C_3}{2} \qquad (2)$$

$$y'' = C_1 e^{x} + 4C_2 e^{2x} + \frac{C_3}{4} e^{\frac{x}{2}}$$

$y''=9$, $x=0$:
$$9 = C_1 + 4C_2 + \frac{C_3}{4} \qquad (3)$$

$$\begin{array}{ccc} C_1 & C_2 & C_3 \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & \frac{1}{2} & 5 \\ 1 & 4 & \frac{1}{4} & 9 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 3 & -\frac{3}{4} & 6 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & \frac{3}{4} & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \frac{4}{3} R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$
\begin{array}{l}
R_1 - \frac{3}{2}R_3 \\
R_2 + \frac{1}{2}R_3
\end{array}
\quad
\begin{array}{ccc|c}
c_1 & c_2 & c_3 & \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

$$c_1 = 1$$
$$c_2 = 2$$
$$c_3 = 0$$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{\frac{x}{2}}$$

$$y = e^{x} + 2e^{2x}$$

4.4 Non-homogeneous DE's with Constant Coefficients

$$y'' - 5y' + 6y = x^3$$

$$y = \underbrace{c_1 e^{2x} + c_2 e^{3x}}_{\text{Complementary solution}} + \underset{\text{particular solution}}{\boxed{\quad\quad\quad}}$$