

4.3 Homogeneous DE's with Constant Coefficients

Ex: Solve $y'' - 4y' - 21y = 0$

auxiliary equation $m^2 - 4m - 21 = 0$
 $(m-7)(m+3) = 0$
 $m = 7, -3$

Roots	Solutions
distinct real # m_1, m_2, \dots	$e^{m_1 x}, e^{m_2 x}, \dots$
repeated real # m_1, m_1, \dots	$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \dots$
Complex # $\alpha \pm \beta i$	$e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x$

$$y_1 = e^{7x}$$

$$y_2 = e^{-3x}$$

General Solution

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{7x} + C_2 e^{-3x}$$

Check:

$$\begin{cases} y = C_1 e^{7x} + C_2 e^{-3x} \\ y' = 7C_1 e^{7x} - 3C_2 e^{-3x} \\ y'' = 49C_1 e^{7x} + 9C_2 e^{-3x} \end{cases}$$

$$LS = y'' - 4y' - 21y$$

$$= (49C_1 e^{7x} + 9C_2 e^{-3x}) - 4(7C_1 e^{7x} - 3C_2 e^{-3x}) - 21(C_1 e^{7x} + C_2 e^{-3x})$$

$$= 0 C_1 e^{7x} + 0 C_2 e^{-3x}$$

0 because 7 is a root of $m^2 - 4m - 21$

$$= 0 \\ = RS \checkmark$$

Ex:

Solve $2y'' - 24y' + 72y = 0$

$$2m^2 - 24m + 72 = 0$$

$$m^2 - 12m + 36 = 0$$

$$(m - 6)^2 = 0$$

$$m = 6, 6$$

$$y_1 = e^{6x}$$

$$y_2 = x e^{6x}$$

General Solution

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{6x} + C_2 x e^{6x} \checkmark$$

$$\text{or } y = (C_1 + C_2 x) e^{6x} \checkmark$$

Ex:

Solve $y'' - 6y' + 13y = 0$

$$m^2 - 6m + 13 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2}$$

$$m = \frac{6 \pm \sqrt{-16}}{2} \quad \sqrt{16} \sqrt{-1}$$

$$m = \frac{6 \pm 4i}{2}$$

$$m = 3 \pm 2i \quad (\alpha = 3, \beta = 2)$$

$$y_1 = e^{\alpha x} \cos \beta x \\ = e^{3x} \cos 2x$$

$$y_2 = e^{\alpha x} \sin \beta x \\ = e^{3x} \sin 2x$$

General Solution $y = C_1 y_1 + C_2 y_2$

$$y = C_1 e^{3x} \cos 2x + C_2 e^{3x} \sin 2x \quad \checkmark$$

$$y = e^{3x} (C_1 \cos 2x + C_2 \sin 2x) \quad \checkmark$$

Ex: Solve $y'' + 49y = 0$, $y(0) = -5$, $y'(0) = 21$

$$m^2 + 49 = 0$$

$$m^2 = -49$$

$$m = \pm \sqrt{49} \sqrt{-1}$$

$$m = \pm 7i \quad (\alpha = 0, \beta = 7)$$

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

General Solution $y = C_1 \cos 7x + C_2 \sin 7x$

$$\begin{array}{l} y = -5 \\ x = 0 \end{array} : -5 = C_1$$

$$y = -5 \cos 7x + C_2 \sin 7x$$

$$y' = 35 \sin 7x + 7C_2 \cos 7x$$

$$\begin{array}{l} y' = 21 \\ x = 0 \end{array} : 21 = 7C_2 \\ C_2 = 3$$

$$y = -5 \cos 7x + 3 \sin 7x$$

Ex: Solve $y^{(4)} + 8y'' + 16y = 0$

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2)^2 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$



$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm \sqrt{-4}$$

$$m = \pm 2i$$

$$m = \pm 2i, \pm 2i$$

$$(\alpha = 0, \beta = 2)$$

$$y_1 = e^{\alpha x} \cos \beta x \\ = \cos 2x$$

$$y_2 = e^{\alpha x} \sin \beta x \\ = \sin 2x$$

$$y_3 = x \cos 2x$$

$$y_4 = x \sin 2x$$

General Solution $y = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$

$$y = C_1 \cos 2x + C_2 \sin 2x + C_3 x \cos 2x + C_4 x \sin 2x \checkmark$$

$$y = (C_1 + C_3 x) \cos 2x + (C_2 + C_4 x) \sin 2x \checkmark$$