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4.1 Intro to Higher-Order Linear DE's

n^{th} order linear DE:

$$\underbrace{a_n(x)}\, y^{(n)} + \dots + \underbrace{a_2(x)}\, y'' + \underbrace{a_1(x)}\, y' + \underbrace{a_0(x)}\, y = g(x)$$

Coefficients are functions of x

The DE is <u>homogeneous</u> if $g(x) = 0$

<u>Ex</u>: a) $y'' - 9y = 0$

is a homogeneous 2nd order linear DE

b) $y'' - 9y = x$

is a non-homogeneous 2nd order linear DE

NOT TO BE CONFUSED WITH:

$$x^2\,dx + (3x^2 - y^2)\,dy = 0$$

(Section 2.5) DE is <u>homogeneous of degree 2</u>

<u>Ex</u>:

$$y'' - 9y = xy$$

$$y'' - 9y - xy = 0$$

$$y'' - (9+x)y = 0$$

homogeneous DE

Ex: $y'' - 9y = 0$

has general solution $y = C_1 e^{3x} + C_2 e^{-3x}$ $(C_1, C_2 :$ any real $\#)$

describes all possible solutions

Some particular solutions:

$$y = 2e^{3x}$$
$$y = 4e^{-3x}$$
$$y = 0$$
$$y = \sqrt{2}\, e^{3x} + \sqrt{5}\, e^{-3x}$$

Def

Functions $f(x)$, $g(x)$ and $h(x)$ are linearly dependent exactly when

$$\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = 0$$

the Wronskian

Ex: a) $f(x) = \sin^2 x$, $g(x) = \cos^2 x$, $h(x) = 7$ are linearly dependent because

$$\begin{vmatrix} \sin^2 x & \cos^2 x & 7 \\ 2\sin x\cos x & -2\cos x \sin x & 0 \\ f''(x) & -f''(x) & 0 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ & & + \end{bmatrix}$$

$$= +7 \begin{vmatrix} 2\sin x\cos x & -2\sin x\cos x \\ f''(x) & -f''(x) \end{vmatrix}$$

$$= 7\left[-2\sin x\cos x\, f''(x) + 2\sin x\cos x\, f''(x)\right]$$

$$= 0$$

b) $f(x) = e^{2x}$, $g(x) = e^{3x}$, $h(x) = 1$ are linearly independent because

$$\begin{vmatrix} e^{2x} & e^{3x} & 1 \\ 2e^{2x} & 3e^{3x} & 0 \\ 4e^{2x} & 9e^{3x} & 0 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ & & - \\ & & + \end{bmatrix}$$

$$= 1 \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 1\,(18e^{5x} - 12e^{5x})$$

$$= 6e^{5x}$$

$$\neq 0$$

Fact

To solve a homogeneous n^{th} order linear DE:

Find n linearly independent solutions $y_1, y_2, \dots, y_n$

General solution to DE: $y = C_1y_1 + C_2y_2 + \dots + C_ny_n$

Ex: $y''' - 11y'' + 30y' = 0$ has
3 linearly independent solutions:

$y_1 = e^{5x}$ $y_2 = e^{6x}$ $y_3 = 1$

General solution?

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$y = C_1 e^{5x} + C_2 e^{6x} + C_3$$

DE is a homogeneous 3rd order linear DE

4.2 Finding a Second Solution

Standard form for a homogeneous 2nd order linear DE:

$$y'' + P(x) y' + Q(x) y = 0$$

Given one solution y_1,
a 2nd linearly independent solution is

$$y_2 = y_1 \boxed{\int \frac{e^{-\int P(x)dx}}{y_1^2} dx}$$

formula sheet

Ex: $y'' = 6y' - 9y$

a) Check that $y_1 = e^{3x}$ is a solution

b) Find y_2

c) Find the general solution