3.1 Cont'd

Newton's Law of Cooling

Rate at which an object warms/cools is proportional to the temperature difference between it and its environment.

T = object's temperature

T_m = ambient temperature

t = time

$$\frac{dT}{dt} = k(T - T_m) \qquad k: \text{constant}$$

Ex: Room temp = 20°C
Initially coffee is 90°C. After 40 mins the coffee is 30°C. Time when coffee is 60°C?

T = coffee temp (°C)

t = time (mins)

$$\frac{dT}{dt} = k(T - 20), \quad T(0) = 90, \quad T(40) = 30$$

$$\frac{dT}{T-20} = k\,dt$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln|T-20| = kt + C_1$$

$$|T-20| = \cancel{e^{kt+C_1}} \qquad e^{C_1} e^{kt}$$

$$T-20 = \underbrace{\pm e^{C_1}}_{C} e^{kt}$$

$$\boxed{T = 20 + Ce^{kt}}$$

$t=0$, $T=90$: $\quad 90 = 20 + C$

$$C = 70$$

$$\boxed{T = 20 + 70e^{kt}}$$

$t=40$, $T=30$: $\quad 30 = 20 + 70e^{40k}$

$$10 = 70e^{40k}$$

$$\frac{1}{7} = e^{40k}$$

$$\ln\frac{1}{7} = 40k$$

$k = \frac{1}{40} \ln \frac{1}{7}$

$$\boxed{T = 20 + 70 e^{\left(\frac{1}{40} \ln \frac{1}{7}\right) t}}$$

$T = 60$:

$$60 = 20 + 70 e^{\left(\frac{1}{40} \ln \frac{1}{7}\right) t}$$

$$40 = 70 e^{\left(\frac{1}{40} \ln \frac{1}{7}\right) t}$$

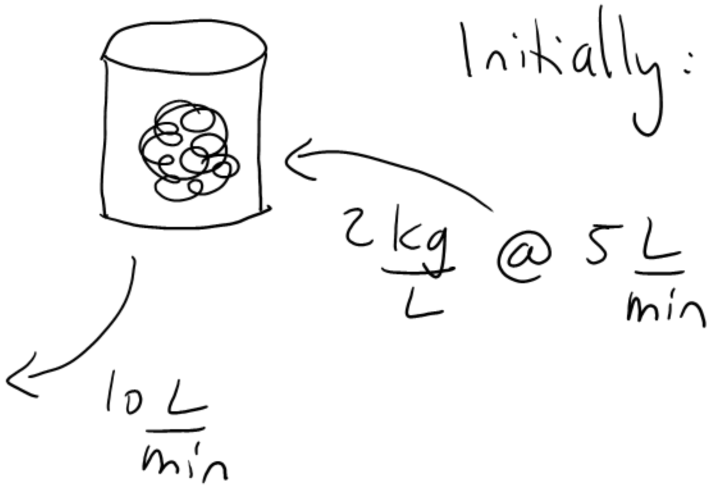
$$\frac{4}{7} = e^{\left(\frac{1}{40} \ln \frac{1}{7}\right) t}$$

$$\ln \frac{4}{7} = \left(\frac{1}{40} \ln \frac{1}{7}\right) t$$

$$\frac{40 \ln\left(\frac{4}{7}\right)}{\ln\left(\frac{1}{7}\right)} = t$$

$t \approx 12$ minutes

Ex: A tank initially contains 500L of water. A solution of 2 kg salt/L flows in at 5L/min. Contents are mixed, and flow out at 10L/min.
Find a formula for the mass of salt in the tank.



Initially: 500L water

$A =$ mass of salt (kg)

$t =$ time (mins)

$A(0) = 0$

$$\frac{dA}{dt} = \text{Inflow Rate} - \text{Outflow Rate} \qquad \left(\frac{kg}{min}\right)$$

$$\boxed{\begin{array}{l}\text{Volume} = 500 - 5t \\ \text{Net loss of } 5 \text{ L/min}\end{array}} \qquad (L)$$

$$\frac{dA}{dt} = 2\,\frac{kg}{L} \cdot 5\,\frac{L}{min} - \frac{A}{500-5t}\,\frac{kg}{L} \cdot 10\,\frac{L}{min}$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t} \qquad \left(\frac{kg}{min}\right)$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10 \qquad \boxed{\text{Linear DE}}$$

$$P(t) = \frac{2}{100-t}$$

$$\text{I.F.} = e^{\int \frac{2}{100-t}\,dt}$$

$$= e^{-2\ln|100-t|}$$

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C}$$

$$= e^{\ln|100-t|^{-2}}$$

$$= |100-t|^{-2}$$

$$= (100-t)^{-2}$$

Interval: $t < 100$

Standard Form: $\frac{dA}{dt} + \frac{2}{100-t}A = 10$

$$(100-t)^{-2}\frac{dA}{dt} + 2(100-t)^{-3}A = 10(100-t)^{-2}$$

Integrate w.r.t. t:

$$(100-t)^{-2}A = 10(100-t)^{-1} + C_1$$

$$\boxed{\begin{aligned} &\int 10(100-t)^{-2}\,dt \\ &= \int -10u^{-2}\,du \\ &= 10u^{-1} + C_1 \end{aligned}}$$

$$A = 10(100-t)^1 + C_1(100-t)^2$$

$A = 0$, $t = 0$: $\quad 0 = 10(100) + C_1(10{,}000)$

$$C_1 = -0.1$$

$$A = 10(100-t) - 0.1(100-t)^2 \quad \checkmark$$

or $\quad A = 1000 - 10t - 0.1(10{,}000 - 200t + t^2)$

$$= 1000 - 10t - 1000 + 20t - 0.1t^2$$

$$= 10t - 0.1t^2 \quad \checkmark$$

Quick Ex : $\quad A(20) = 160$ kg