2.3 Linear DE's Cont'd

Ex: Solve explicitly:

$$t\frac{dy}{dt} + 3y = t+1, \quad y(-1) = \frac{1}{3}$$

$$\boxed{\frac{dy}{dt} + \frac{3}{t}y = 1 + \frac{1}{t}}$$

$$P(t) = \frac{3}{t}$$

$$\text{I.F.} = e^{\int \frac{3}{t}\,dt}$$

$$= e^{3\ln|t|} \quad \leftarrow \text{no constant}$$

$$= e^{\ln|t|^3}$$

$$= |t|^3$$

$$= \begin{cases} t^3, & t>0 \\ -t^3, & t<0 \end{cases}$$

$$\text{I.F.} = -t^3 \quad (\text{Interval of solution: } t<0)$$

$$-t^3\frac{dy}{dt} - 3t^2 y = -t^3 - t^2$$

Integrate w.r.t. t:

$$-t^3 y = -\frac{t^4}{4} - \frac{t^3}{3} + C_1$$

$$y = \frac{t}{4} + \frac{1}{3} + \frac{C_2}{t^3}$$

$y = \frac{1}{3}$, $t=-1$:

$$\frac{1}{3} = \frac{-1}{4} + \frac{1}{3} - C_2$$

$$C_2 = \frac{-1}{4}$$

$$y = \frac{t}{4} + \frac{1}{3} - \frac{1}{4t^3}$$

Ex: Solve explicitly:

$$y\,dx - (x+y^6)\,dy = 0$$

$$y - (x+y^6)\frac{dy}{dx} = 0$$

$$\frac{-1}{x+y^6}\,y + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - \frac{1}{x+y^6}\,y = 0$$ Not linear in y

$$y\,dx - (x+y^6)\,dy = 0$$

$$y\frac{dx}{dy} - x - y^6 = 0$$

$$\frac{dx}{dy} - \frac{1}{y}x - y^5 = 0$$

$$\boxed{\frac{dx}{dy} - \frac{1}{y}x = +y^5}$$ Linear in x

$$P(y) = \frac{-1}{y}$$

$$\text{I.F.} = e^{\int -\frac{1}{y}dy}$$
$$= e^{-\ln|y|}$$
$$= e^{\ln|y|^{-1}}$$
$$= |y|^{-1}$$
$$= y^{-1} \quad (y>0)$$

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = +y^4$$

Integrate w.r.t. y:

$$\frac{1}{y}x = +\frac{y^5}{5} + C_1$$

$x = +\frac{y^6}{5} + Cy$

2.4 Exact DE's

Let $f = x^3 y + \sin(3x+y)$

Partial Derivatives:

$$\frac{\delta f}{\delta x} = \frac{\partial f}{\partial x} = f_x = 3x^2 y + 3\cos(3x+y)$$

$$\frac{\delta f}{\delta y} = \frac{\partial f}{\partial y} = f_y = x^3 + \cos(3x+y)$$

The differential of f is $df = f_x\,dx + f_y\,dy$

Here $df = [3x^2 y + 3\cos(3x+y)]\,dx + [x^3 + \cos(3x+y)]\,dy$

A DE $M\,dx + N\,dy = 0$ is exact if the left side is a differential.

Quick Way to Check if a DE is exact:

$M_y = N_x \Rightarrow$ DE is exact

Ex: $(3x^2 + \sin y)\,dx + x\cos y\,dy = 0$

$M = 3x^2 + \sin y$ $\quad N = x\cos y$

$M_y = \cos y$ $\quad N_x = \cos y$

$M_y = N_x \Rightarrow$ DE is exact

Ex: $\underbrace{x^3 y}_{M}\,dx + \underbrace{(x+1)}_{N}\,dy = 0$

$M_y = x^3 \qquad N_x = 1$

$M_y \neq N_x \Rightarrow$ DE is not exact

$\Rightarrow$ LS of DE is <u>not</u> a differential

Ex: Solve $\underbrace{\sin y}_{M}\, dx + \underbrace{(2y + x\cos y)}_{N}\, dy = 0$

$M_y = \cos y \qquad N_x = \cos y$

$M_y = N_x \Rightarrow$ DE is exact

DE: $df = 0$

$\int df = \int 0$

Solution: $f = C$

$f = \int \sin y \, dx$ AND $f = \int (2y + x\cos y)\, dy$

$= x\sin y + g(y)$ (y-terms) $\qquad = y^2 + x\sin y + h(x)$ (x-terms)

$g(y) = y^2$

$h(x) = 0$

$\Rightarrow f = x\sin y + y^2$

DE: $df = 0$

$\int df = \int 0$

Solution: $f = C$

$\boxed{x\sin y + y^2 = C}$