2.3 Linear DE's

We'll solve $\frac{dy}{dx} + P(x)\,y = f(x)$

"1st order linear DE"

Recall: Product Rule

$$\frac{d}{dx}\left[e^{x^2} y \right] = e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y$$

$$\int \left(e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y \right) dx = e^{x^2}y + \text{constant}$$

shortcut

<u>Ex</u>: Solve explicitly (solve for y)

$$y' + 2xy - x = 0$$

1) Standard Form $\frac{dy}{dx} + P(x)y = f(x)$

$$\frac{dy}{dx} + 2xy = x$$

2) $P(x) = 2x$

3) Integrating Factor

$$I.F. = e^{\int P(x)\,dx}$$

$$= e^{\int 2x\,dx}$$

$$= e^{x^2} \leftarrow \text{Don't use a constant}$$

4) Multiply the standard form by the I.F.

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

5) Integrate with respect to x
Left side comes from the product rule
Use the shortcut for the left side.

$$\int x e^{x^2} dx$$
$$u = x^2$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$
$$\int \frac{1}{2} e^u du$$
$$= \frac{1}{2} e^u + C_1$$
$$= \frac{1}{2} e^{x^2} + C_1$$

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C_1$$

6) Explicit Solution
Use Initial Conditions, if applicable.

$$y = \frac{1}{2} + Ce^{-x^2}$$

Ex: Solve explicitly

$$x\frac{dy}{dx} - y - 2x^2 = 0 \quad , \quad y(1) = 5$$

$$\frac{dy}{dx} - \frac{1}{x}y - 2x = 0$$

$$\boxed{\frac{dy}{dx} - \frac{1}{x}y = 2x}$$

$$P(x) = -\frac{1}{x}$$

$$I.F. = e^{\int -\frac{1}{x} dx}$$
$$= e^{-\ln|x|} \quad \leftarrow \text{no constant}$$
$$= e^{\ln|x|^{-1}}$$

$$= |x|^{-1}$$

$$= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{-1}{x}, & x < 0 \end{cases}$$

$= \frac{1}{x}$ because initial condition was for $x = 1$

(Interval: $x > 0$)

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2$$

Integrate w.r.t. x :

$$\frac{1}{x}y = 2x + C$$

$$\boxed{y = 2x^2 + Cx}$$

$y = 5$, $x = 1$: $5 = 2 + C$

$$C = 3$$

$$\boxed{y = 2x^2 + 3x}$$

(Interval: $x > 0$)

Ex: Solve explicitly

$$(x^2 - 16)\frac{dy}{dx} + xy = x$$

$$\boxed{\frac{dy}{dx} + \frac{x}{x^2 - 16}y = \frac{x}{x^2 - 16}}$$

$$P(x) = \frac{x}{x^2 - 16}$$

$$\text{I.F.} = e^{\int \frac{x}{x^2 - 16}dx}$$

$u = x^2 - 16$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$$\int \frac{x}{x^2-16}\,dx$$
$$= \int \frac{1}{2}\,\frac{du}{u}$$
$$= \frac{1}{2}\ln|u| + \text{Constant}$$

$$\text{I.F.} = e^{\frac{1}{2}\ln|x^2-16|}$$
$$= e^{\ln|x^2-16|^{1/2}}$$
$$= \sqrt{|x^2-16|}$$
$$= \sqrt{x^2-16}$$

$$\sqrt{x^2-16}\,\frac{dy}{dx} + \frac{x}{\sqrt{x^2-16}}\,y = \frac{x}{\sqrt{x^2-16}}$$

Integrate w.r.t. x

$$u = x^2-16$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$
$$\int \frac{x}{\sqrt{x^2-16}}\,dx$$
$$= \int \frac{1}{2}\,\frac{du}{\sqrt{u}}$$
$$= \frac{1}{2}\left(2u^{1/2}\right) + C$$

$$\sqrt{x^2-16}\;y = \sqrt{x^2-16} + C$$
$$y = 1 + \frac{C}{\sqrt{x^2-16}}$$

Note: $\dfrac{C}{\sqrt{x^2-16}}$ is a transient term

because $\dfrac{C}{\sqrt{x^2-16}} \to 0$ as $x \to \infty$

$$\lim_{x\to\infty} y \;?$$

$$\lim_{x\to\infty} y = 1$$

Transient terms:

$$\frac{1}{x}$$

$$\frac{1}{\sqrt{x}}$$

$$\frac{1}{\ln x}$$

$$\frac{c}{\ln x}$$

Non-transient terms:

$$2$$

$$x$$

$$\sin x$$

$$\frac{1}{\sin x}$$