(10) Cont'd

$\vdots$

$$s^2 Y(s) - 1 + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s}\frac{s}{s^2+4}$$

$$(s^2+4)Y(s) = 1 + \frac{s}{s^2+4} - e^{-\pi s}\frac{s}{s^2+4}$$

$$Y(s) = \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2} - e^{-\pi s}\frac{s}{(s^2+4)^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{2}\frac{1}{(s^2+4)} + \frac{4}{4}\frac{s}{(s^2+4)^2} - e^{-\pi s}\frac{4}{4}\frac{s}{(s^2+4)^2}\right\}$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - f(t-\pi)\mathcal{U}(t-\pi)$$

$$\boxed{\begin{aligned} F(s) &= \frac{4}{4}\frac{s}{(s^2+4)^2} \\ f(t) &= \frac{1}{4}t\sin 2t \\ f(t-\pi) &= \frac{1}{4}(t-\pi)\sin[2(t-\pi)] \end{aligned}}$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - \frac{1}{4}(t-\pi)\sin[2(t-\pi)]\mathcal{U}(t-\pi)$$

(11)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 6 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 24 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda-8)(\lambda+2) = 0$$

$$\lambda = 8, -2$$

$\lambda = 8:$

$$[A - 8I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} -6 & 4 & 0 \\ 6 & -4 & 0 \end{array}\right]$$

$\frac{R_1}{-6}$ ...

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = t$

$$x_1 - \frac{2}{3} x_2 = 0 \implies x_1 = \frac{2}{3} t$$

$$\vec{x} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} t$$

$$\vec{K} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\lambda = -2:$

$$[A + 2I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 6 & 6 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$$x_1 + x_2 = 0 \implies x_1 = -t$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

$$\vec{K} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{X} = C_1 \vec{K} e^{\lambda t} + C_2 \vec{K} e^{\lambda t}$$

$$\vec{X} = C_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{8t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

## Practice Questions

(29) b) Find $\mathcal{L}^{-1}\left\{ \dfrac{e^{-2s}}{s+3} \right\}$

$$= f(t-2)\,\mathcal{U}(t-2)$$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{s+3} \\ f(t) &= e^{-3t} \\ f(t-2) &= e^{-3(t-2)} \end{aligned}}$$

$$= e^{-3(t-2)}\,\mathcal{U}(t-2)$$

d) Find $\mathcal{L}^{-1}\left\{ \dfrac{s+4}{s^2+2s+5} \right\}$

$$\begin{aligned} s^2+2s+5 &= (s+1)^2 + \,? \\ &= (s+1)^2 + 4 \\ &= (s+1)^2 + 2^2 \end{aligned}$$

$$\begin{aligned} s+4 &= \,?\,(s+1) + \,? \\ &= (s+1) + \,? \\ &= (s+1) + 3 \end{aligned}$$

$$\mathcal{L}^{-1}\left\{ \frac{s+4}{s^2+2s+5} \right\} = \mathcal{L}^{-1}\left\{ \frac{(s+1)+3}{(s+1)^2+2^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\,\frac{2}{(s+1)^2+2^2} \right\}$$

$$= e^{-t}\cos 2t + \frac{3}{2} e^{-t} \sin 2t$$

e) $\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\}$

$$= e^{-\pi s}\mathcal{L}\{f(t+\pi)\}$$

$$\boxed{\begin{aligned} f(t) &= \cos t \\ f(t+\pi) &= \cos(t+\pi) \\ &= \cos t\cos\pi - \sin t\sin\pi \\ &= -\cos t \end{aligned}}$$

$$= e^{-\pi s}\mathcal{L}\{-\cos t\}$$

$$= -e^{-\pi s}\frac{s}{s^2+1}$$

30 Solve using $\mathcal{L}$:

$$y'' + 2y' + 5y = \cos t, \quad y(0)=2, \quad y'(0) = -1$$

1) Apply $\mathcal{L}$ and 2) Solve for $Y(s)$

⋮

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+5)} + \frac{2s+3}{s^2+2s+5}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+5)} + \frac{2s+3}{s^2+2s+5}\right\} \quad (*)$$

$$\frac{s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$s = (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$$

$s = i$ :

$$i = (Ai+B)(4+2i)$$

$$i = 4Ai - 2A + 4B + 2Bi$$

$$i = (4A+2B)i + (-2A+4B)$$

$$4A+2B = 1 \quad (1)$$

$$\cancel{-2A+4B=0} \quad (2)$$

$$+ \quad -4A+8B=0$$

$$10B = 1 \Rightarrow B = \frac{1}{10}$$

$$B = \frac{1}{10} \rightarrow (2) : \quad A = \frac{1}{5}$$

$s^3$ coefficient: $\quad 0 = A + C \Rightarrow C = -\frac{1}{5}$

Constant : $\quad 0 = 5B + D \Rightarrow D = -\frac{1}{2}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{5}s + \frac{1}{10}}{s^2+1} + \frac{-\frac{1}{5}s - \frac{1}{2}}{s^2+2s+5} + \frac{2s+3}{s^2+2s+5} \right\}$$