8 a) ⋮

$x'' + 9x = 0 \quad , \quad x(0) = 2, \quad x'(0) = -1$

$m^2 + 9 = 0$

$m^2 = -9$

$m = \pm \cancel{\sqrt{-9}}\ 3i \qquad (\alpha = 0 \quad \beta = 3)$

$\cancel{y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)}$

$x = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$

$x = C_1 \cos 3t + C_2 \sin 3t$

$t=0$, $x=2$: $\quad 2 = C_1(1) + C_2(0)$

$C_1 = 2$

$\boxed{x = 2\cos 3t + C_2 \sin 3t}$

$x' = -6\sin 3t + 3C_2 \cos 3t$

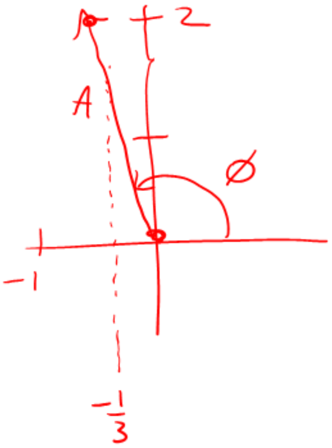
$x' = -1$, $t=0$: $\quad -1 = -6(0) + 3C_2(1)$

$3C_2 = -1$

$C_2 = -\frac{1}{3}$

$x = 2\cos 3t - \frac{1}{3}\sin 3t$

b) $x = \underset{A\sin\phi}{2}\cos 3t \underset{A\cos\phi}{-\frac{1}{3}}\sin 3t$



$$A = \sqrt{\left(-\tfrac{1}{3}\right)^2 + (2)^2}$$

$$= \sqrt{\tfrac{1}{9} + 4}$$

$$= \sqrt{\tfrac{37}{9}}$$

$$= \frac{\sqrt{37}}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{\left(-\frac{1}{3}\right)}\right) \quad (+\pi\,?)$$

$$= \tan^{-1}(-6) + \pi$$

$$\approx 1.74$$

$$x = A\sin(\omega t + \phi)$$

$$= \frac{\sqrt{37}}{3}\sin(3t + 1.74)$$

9

$$(x+2)y'' + y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$(x+2)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\cancel{x}\sum_{n=2}^{\infty} n(n-1) c_n \cancel{x^{n-2}}\, x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$k=n-1$
$n=k+1$
$n=2 \Rightarrow k=1$

$k=n-2$
$n=k+2$
$n=2 \Rightarrow k=0$

$k=n$
$n=0 \Rightarrow k=0$

Start at largest k : $k=1$

$$\sum_{k=1}^{\infty} \quad + (1^{st} \text{ term}) + \sum_{k=1}^{\infty} \quad + (1^{st} \text{ term}) + \sum_{k=1}^{\infty} \quad = 0$$

$$\sum_{k=1}^{\infty} (k+1)kC_{k+1} x^k + 4C_2 x^0 + \sum_{k=1}^{\infty} 2(k+2)(k+1)C_{k+2} x^k$$

$$+ C_0 x^0 + \sum_{k=1}^{\infty} C_k x^k = 0$$

$$(4C_2 + C_0) + \sum_{k=1}^{\infty} \left[(k+1)kC_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k\right] x^k = 0$$

$$4C_2 + C_0 = 0 \Rightarrow C_2 = -\frac{C_0}{4}$$

$$(k+1)kC_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k = 0$$

$$C_{k+2} = \frac{-(k+1)kC_{k+1} - C_k}{2(k+2)(k+1)}, \quad k \geq 1$$

$(k=1)$

$$C_3 = \frac{-2C_2 - C_1}{12}$$

$$= -\frac{C_2}{6} - \frac{C_1}{12}$$

$$= \frac{C_0}{24} - \frac{C_1}{12}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$= C_0 + C_1 x - \frac{C_0}{4} x^2 + \left(\frac{C_0}{24} - \frac{C_1}{12}\right) x^3 + \dots$$

$$= C_0 \underbrace{\left[1 - \frac{x^2}{4} + \frac{x^3}{24} + \dots\right]}_{y_1} + C_1 \underbrace{\left[x - \frac{x^3}{12} + \dots\right]}_{y_2}$$

Aside

$$y(0) = 9$$

$$y'(0) = 13$$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$9 = C_0$$

$$y' = C_1 + 2C_2 x + \dots$$

$$13 = C_1$$

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$$f(t) = \begin{cases} \cos 2t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

$$f(t) = \underset{\text{Initial}}{\square} + \underset{\text{Change}}{\square}\,\mathcal{U}(t-\pi)$$

$$= \cos 2t - \cos 2t\,\mathcal{U}(t-\pi)$$

$$y'' + 4y = \cos 2t - \cos 2t\, \mathcal{U}(t-\pi)$$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s)$$

$$= \frac{s}{s^2+4} - e^{-\pi s} \underbrace{\mathcal{L}\{g(t+\pi)\}}$$

$g(t) = \cos 2t$

$$\mathcal{L}\{\cos 2(t+\pi)\}$$
$$= \mathcal{L}\{\cos(2t+2\pi)\}$$
$$= \mathcal{L}\{\cos 2t\}$$
$$= \frac{s}{s^2+4}$$

$$s^2 Y(s) - 1 + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s}\frac{s}{s^2+4}$$