1st order DE

Separable

Linear / Bernoulli

Homogeneous Degree

Sub $u = Ax + By + C$

Exact

Make Exact with I.F.

① $M\,dx + N\,dy = 0$

$M = y + y\cos xy$

$M_y = 1 + \cos xy - xy\sin xy$

$N = x + x\cos xy + \frac{2}{y}$

$N_x = 1 + \cos xy - xy\sin xy$

$M_y = N_x$

$\Rightarrow$ DE is exact

$f = \int (y + y\cos xy)\,dx$ AND $f = \int \left(x + x\cos xy + \frac{2}{y}\right)dy$

$= xy + \sin xy + g(y)$ $\quad = xy + \sin xy + 2\ln|y| + h(x)$

$\Rightarrow f = xy + \sin xy + 2\ln|y|$

DE: $df = 0$

Solution: $f = C$

$xy + \sin xy + 2\ln|y| = C$

② $y' + \frac{y}{x} = x^3 y^3$ Bernoulli $n = 3$

$$\begin{cases} y = u^{\frac{1}{1-n}} = u^{-1/2} \\ \frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx} \end{cases}$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx} + \frac{1}{x}u^{-1/2} = x^3 u^{-3/2}$$

Mult. by $-2u^{3/2}$:

$$\frac{du}{dx} - \frac{2}{x}u = -2x^3$$

Linear ☺

$$P(x) = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\ln|x|}$$

$$= e^{\ln|x|^{-2}}$$

$$= |x|^{-2}$$

$$= x^{-2}$$

$$x^{-2}\frac{du}{dx} - 2x^{-3}u = -2x$$

$$x^{-2}u = -x^2 + C$$

$$\boxed{\begin{aligned} y &= u^{-1/2} \\ y^{-2} &= u \end{aligned}}$$

$$x^{-2}y^{-2} = -x^2 + c$$

(3) $(x^2+y^2)dx - xy\,dy = 0$

Homogeneous (of Degree 2)

$$\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$$

$$(x^2 + u^2x^2)\,dx - \underbrace{\cancel{x(ux)}}_{ux^2}(u\,dx + x\,du) = 0$$

$$x^2dx + u^2x^2dx - u^2x^2dx \quad - ux^3du = 0$$

Separable

$$x^2dx - ux^3du = 0$$

$$x^2\,dx = ux^3\,du$$

$$\frac{dx}{x} = u\,du$$

$$\ln|x| = \frac{u^2}{2} + C_1$$

$$\boxed{\begin{aligned} y &= ux \\ u &= y/x \end{aligned}}$$

$$\ln|x| = \frac{1}{2}\frac{y^2}{x^2} + C_1$$

$$2x^2\ln|x| = y^2 + 2C_1x^2$$

$$y^2 = 2x^2\ln|x| \underbrace{\cancel{-2C_1}}_{+C_2}x^2$$

$$y = \pm\sqrt{2x^2\ln|x| + C\,x^2}$$

④

$$ma \propto -v$$

$$ma = -kv$$

$$\boxed{a = \frac{dv}{dt}}$$

$$m\frac{dv}{dt} = -kv$$

Separable
(or Linear)

$$\frac{m\,dv}{v} = -k\,dt$$

$$\frac{dv}{v} = -\frac{k}{m}\,dt$$

$$\ln|v| = -\frac{k}{m}t + C_1$$

$$e^{\ln|v|} = e^{-\frac{k}{m}t + C_1}$$

$$|v| = e^{-\frac{k}{m}t} \cdot e^{C_1}$$

$$v = \underbrace{\cancel{\pm e^{C_1}}}_{C}\, e^{-\frac{k}{m}t}$$

$v = v_0$
$t = 0$:

$$v_0 = C$$

$$v = v_0\, e^{-\frac{k}{m}t}$$