8.2 Cont'd

Case 3: Repeated λ with not enough eigenvectors

Specifically: algebraic multiplicity of $\lambda > 1$

and geometric " $= 1$

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t}$$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{P}) e^{\lambda t} \quad \text{where} \quad [A - \lambda I]\vec{P} = \vec{K}_1$$

$$\vec{X}_3 = \left(\vec{K}_1 \frac{t^2}{2} + \vec{P}t + \vec{Q}\right) \quad \text{where} \quad [A - \lambda I]\vec{Q} = \vec{P}$$

If λ is repeated twice, use $\vec{X}_1$ and $\vec{X}_2$

" 3 times, use $\vec{X}_1, \vec{X}_2, \vec{X}_3$

<u>Ex</u>: Solve $\vec{X}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{X}$ given $\lambda = 1, 1$

$\lambda = 1$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A - I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$x_2 = a$$

$$x_1 - 2x_2 = 0 \implies x_1 = 2a$$

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} a$$

$$\vec{K}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{P}$:

$$[A - \lambda I]\vec{P} = \vec{K}_1$$

$$\left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cc} P_1 & P_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$p_2 = a$$

$$p_1 - 2p_2 = 1 \implies p_1 = 1 + 2a$$

$$\vec{p} = \begin{bmatrix} 1+2a \\ a \end{bmatrix}$$

Choose any nonzero $\vec{p}$: $\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{P}) e^{\lambda t} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$$

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$

$$= C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$$

Ex: Solve $\vec{X}' = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix} \vec{X}$

$\lambda = 3, 3, 3$ (A is lower triangular)

$\lambda = 3$: $[A - 3I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

$$x_3 = a$$

$$x_1 = 0$$
$$x_2 = 0$$

$$\vec{K}_1 = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

Choose any nonzero $\vec{K}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{P}$: $[A - \lambda I] \vec{P} = \vec{K}_1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$p_3 = a$$

$$p_1 = 0$$

$$p_2 = 1$$

$$\vec{P} = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}$$

Choose any nonzero $\vec{P}$: $\vec{P} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{Q}$: $[A - \lambda I]\vec{Q} = \vec{P}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$q_3 = a$$

$$q_2 = -2$$

$$q_1 = 1$$

$$\vec{Q} = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}$$

Choose any nonzero $\vec{Q}$: $\vec{Q} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{P})e^{\lambda t} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) e^{3t}$$

$$\vec{X}_3 = \left(\vec{K}_1 \frac{t^2}{2} + \vec{P} t + \vec{Q}\right) e^{\lambda t} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right) e^{3t}$$

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2 + c_3 \vec{X}_3$$

<u>Case 4</u>: $\lambda = \alpha \pm \beta i$

Find the eigenvector $\vec{K}_1$ for $\lambda = \alpha + \beta i$

Let $\vec{B}_1 = \mathrm{Re}(\vec{K}_1)$ $\quad \vec{B}_2 = \mathrm{Im}(\vec{K}_1)$

Ex: $\vec{K}_1 = \begin{bmatrix} 2i \\ 3-4i \end{bmatrix}$

$\vec{B}_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\vec{B}_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

See formula sheet:

$$\vec{X}_1 = e^{\alpha t}\left[\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t\right]$$

$$\vec{X}_2 = e^{\alpha t}\left[\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t\right]$$

Ex: Solve $\vec{X}' = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\vec{X}$

λ:

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 + 4 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{-16}}{2} \Leftarrow \sqrt{16}\sqrt{-1}$$

$$\lambda = \frac{6 \pm 4i}{2}$$

$$\lambda = 3 \pm 2i \qquad (\alpha = 3,\ \beta = 2)$$

$\lambda = 3+2i$:

$$[A - \lambda I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 3-(3+2i) & -2 & 0 \\ 2 & 3-(3+2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

then $R_1 \leftrightarrow R_2$, $\frac{R_1}{2}$

$$\left[\begin{array}{cc|c} 1 & -i & 0 \\ -2i & -2 & 0 \end{array}\right]$$

$$R_2 + 2iR_1 \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{OR} \quad \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$$\boxed{\begin{aligned} &-2 + 2i(-i) \\ &= -2 + 2 \end{aligned}}$$

observe the two rows are multiples (otherwise no nontrivial solution).