Assignment due Tues Apr. 11 8:30am

8.2 Cont'd

Case 1: Distinct Real Eigenvalues

Ex: Solve $\vec{x}' = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$

λ: $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Cofactor Expansion $\begin{bmatrix} + & - & + \\ - & + & \\ + & & \ddots \end{bmatrix}$

Bottom row:

$$(1-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)^2 - 4 \right] = 0$$

$$(1-\lambda)(\lambda^2 - 8\lambda + 12) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-6) = 0$$

$$\lambda = 1, 2, 6$$

$\lambda = 1$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A - I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{R_1}{3}$ $$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 + 2R_1$ $$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{R_2}{\left(\frac{5}{3}\right)} \quad \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + \tfrac{2}{3}R_2 \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF} \longleftarrow \text{no info}$$

$$\boxed{x_3 = a}$$

$x_1 = 0$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a$$

$$\vec{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 2$:

$$[\, A - 2I \mid \vec{0} \,]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = a$

$x_1 - x_2 = 0 \Rightarrow x_1 = a$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} a$$

$$\vec{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 6$:

$$[\, A - 6I \mid \vec{0} \,]$$

$$\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = a$

$x_1 + x_2 = 0 \Rightarrow x_1 = -a$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} a$$

$$\vec{K}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ etc.}$$

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

$$\vec{X} = C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{6t}$$

Case 2: Repeated λ with Enough Eigenvectors

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

Ex: Solve $\quad \vec{X}' = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{X}$

$$\lambda = 2, 2, 1$$

Recall: If A is diagonal / upper triangular / lower triangular then λ = diagonal entries.

$\lambda = 1$: $\quad [A - I \mid \vec{0}]$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_3 = a$

$x_1 + 3x_3 = 0 \Rightarrow x_1 = -3a$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} a \qquad \vec{K}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2:$ $\quad [A - 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_1 = a$

$x_2 = b$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} b$$

$$\vec{K}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{K}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{X} = \sum c_i \vec{K}_i e^{\lambda_i t}$$

$$\vec{X} = c_1 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Case 3: Repeated λ with not enough eigenvectors

Specifically: $\dim(E_\lambda) = 1$ and algebraic multiplicity of $\lambda > 1$

$\vec{X}_1 = \vec{K}_1 e^{\lambda t}$

$\vec{X}_2 = (\vec{K}_1 t + \vec{P}) e^{\lambda t}$ where $[A - \lambda I]\vec{P} = \vec{K}_1$

$\vec{X}_3 = \left(\vec{K}_1 \frac{t^2}{2} + \vec{P} t + \vec{Q}\right) e^{\lambda t}$ where $[A - \lambda I]\vec{Q} = \vec{P}$

If λ is repeated twice, use $\vec{X}_1$ and $\vec{X}_2$

" three times, use $\vec{X}_1$, $\vec{X}_2$ and $\vec{X}_3$