4.6, 4.7, 5.1, 6.2

Ex: Use sigma notation to write $C_2, C_3, \ldots, C_7$ in terms of C_0 or C_1.

$$y'' - 2x^2 y' = 0$$

$$y = \sum_{n=0}^{\infty} C_n x^n \qquad y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} n C_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=1}^{\infty} 2n C_n x^{n+1} = 0$$

$k = n-2$ $\qquad$ $k = n+1$

$n = k+2$ $\qquad$ $n = k-1$

$n=2 \Rightarrow k=0$ $\qquad$ $n=1 \Rightarrow k=2$

Start at $k=2$

$$1^{st} \text{ two terms} + \sum_{k=2}^{\infty} \qquad - \sum_{k=2}^{\infty} \qquad = 0$$

$$2C_2 + 6C_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) C_{k+2} x^k - \sum_{k=2}^{\infty} 2(k-1) C_{k-1} x^k = 0$$

$$2C_2 + 6C_3 x + \sum_{k=2}^{\infty} \left[(k+2)(k+1) C_{k+2} - 2(k-1) C_{k-1}\right] x^k = 0$$

Set all coefficients $= 0$

$2C_2 = 0 \Rightarrow C_2 = 0$

$6C_3 = 0 \Rightarrow C_3 = 0$

$$(k+2)(k+1) C_{k+2} - 2(k-1) C_{k-1} = 0 \Rightarrow C_{k+2} = \frac{2(k-1) C_{k-1}}{(k+2)(k+1)}, \quad k \geq 2$$

$(k=2) \qquad C_4 = \dfrac{2C_1}{4\cdot 3} = \dfrac{C_1}{6}$

$(k=3) \qquad C_5 = \dfrac{4C_2}{5\cdot 4} = 0$

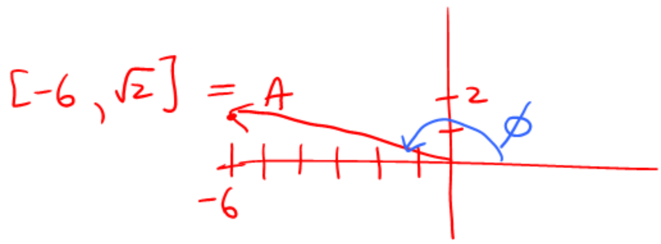
$(k=4) \qquad C_6 = \dfrac{6C_3}{6\cdot 5} = 0$

$(k=5) \qquad C_7 = \dfrac{8C_4}{7\cdot 6} = \dfrac{8}{42}\left(\dfrac{C_1}{6}\right) = \dfrac{2C_1}{63}$

<u>Ex</u>: Find the 2nd time when the mass passes through equilibrium position:

$$x = \underbrace{-6}\sin\frac{t}{2} + \underbrace{\sqrt{2}}\cos\frac{t}{2}$$

Shortcut $\qquad A\cos\phi = -6 \qquad A\sin\phi = \sqrt{2}$



$$A = \sqrt{36+2} = \sqrt{38}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{2}}{-6}\right)(+\ \pi?)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}}{-6}\right) + \pi$$

$$\approx 2.91 \text{ rads}$$

$$x = A\sin\left(\frac{t}{2} + \phi\right) \qquad *$$

$$x = \sqrt{38}\sin\left(\frac{t}{2} + 2.91\right)$$

equilibrium position $\implies x = 0$

$\implies$ angle $= 0, \pi, 2\pi, \ldots$

$\frac{t}{2} + 2.91 = 0, \pi, 2\pi, \ldots$

$\frac{t}{2} + 2.91 = 0$ nonsense

$\frac{t}{2} + 2.91 = \pi \Rightarrow t \approx 0.46\text{ s}$

$\frac{t}{2} + 2.91 = 2\pi \Rightarrow \boxed{t \approx 6.75\text{ s}}$

2nd time

<u>Ex</u>: Solve using variation of parameters:

$$x^2 y'' - 12xy' + 12y = x,$$

given $y_c = C_1 x + C_2 x^{12}$.

1) $y_c = C_1 x + C_2 x^{12}$

$y_1 = x \qquad y_2 = x^{12}$

2) $W = \begin{vmatrix} x & x^{12} \\ 1 & 12x^{11} \end{vmatrix} = 11x^{12}$

$W_1 = \begin{vmatrix} 0 & x^{12} \\ f(x) & 12x^{11} \end{vmatrix}$

$= \begin{vmatrix} 0 & x^{12} \\ \frac{1}{x} & \sim \end{vmatrix}$

$= -x^{11}$

Caution: Standard Form for $f(x)$

$y'' - \ldots = \frac{1}{x} \leftarrow f(x)$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x} \end{vmatrix}$

$= 1$

3) $u_1' = \frac{W_1}{W}$

$= \frac{-x^{11}}{11x^{12}}$

$= \frac{-1}{11} \frac{1}{x}$

$u_1 = -\frac{1}{11} \ln|x|$ No Constant

4) $u_2' = \frac{W_2}{W}$

$= \frac{1}{11 x^{12}}$

$= \frac{1}{11} x^{-12}$

$u_2 = \frac{1}{11}\left(\frac{-1}{11} x^{-11}\right)$ No Constant

$= \frac{-1}{121} x^{-11}$

5) $y_p = u_1 y_1 + u_2 y_2$

$= -\frac{1}{11} x \ln|x| - \frac{1}{121} \cancel{x^{-11}(x^{12})}_{x}$

6) $y = y_c + y_p$

$y = C_1 x + C_2 x^{12} - \frac{1}{11} x \ln|x| - \frac{1}{121} x$ ✓

or $y = C_1 x + C_2 x^{12} - \frac{1}{11} x \ln|x|$ ✓

~~7) Initial Conditions~~

Ex: Solve $x^2 y'' - x y' + 2y = 0$

Cauchy-Euler (as opposed to Constant Coefficient)

$m(m-1) - m + 2 = 0$

$m^2 - m - m + 2 = 0$

$m^2 - 2m + 2 = 0$

$\vdots$

$m = 1 \pm i$ $(\alpha = 1 \quad \beta = 1)$

$$y = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$y = x\left[C_1 \cos(\ln x) + C_2 \sin(\ln x)\right]$$