7.3 Cont'd

$$y(t) = \left[\frac{2}{5} - \frac{2}{5} e^{-5(t-3)} \right] \mathcal{U}(t-3)$$

Follow-Up : Find $y(0), y(1), y(3), y(4)$

$$y(t) = \begin{cases} 0 & 0 \le t < 3 \\ \frac{2}{5} - \frac{2}{5} e^{-5(t-3)}, & t \geq 3 \end{cases}$$

$y(0) = 0$

$y(1) = 0$

$y(3) = \frac{2}{5} - \frac{2}{5} = 0$

$y(4) = \frac{2}{5} - \frac{2}{5} e^{-5}$

7.4 Convolutions and Periodic Functions

See formula sheet for $\mathcal{L}$ of :

$t \sin \omega t$

$t \cos \omega t$

$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$

Ex: Solve using $\mathcal{L}$:

$x'' + 9x = \cos 3t, \quad x(0) = 0, \quad x'(0) = 1$

1) Apply $\mathcal{L}$

$$s^2 X(s) - sx(0) - x'(0) + 9X(s) = \frac{s}{s^2+9}$$

$$s^2 X(s) - 1 + 9X(s) = \frac{s}{s^2+9}$$

2) Solve for $X(s)$

$$(s^2+9)X(s) = 1 + \frac{s}{s^2+9}$$

$$X(s) = \frac{1}{s^2+9} + \frac{s}{(s^2+9)^2}$$

3) Apply $\mathcal{L}^{-1}$

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+9} + \frac{s}{(s^2+9)^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{3}\frac{3}{s^2+9} + \frac{1}{6}\frac{6s}{(s^2+9)^2} \right\}$$

$$= \frac{1}{3}\sin 3t + \frac{1}{6} t \sin 3t$$

Convolution of f and g:

$$f * g = \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

Note: In Sugg HW this is written

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\boxed{\begin{aligned} \mathcal{L}\{f*g\} &= F(s)\,G(s) \\ \mathcal{L}^{-1}\{F(s)G(s)\} &= f*g \end{aligned}}$$

__Quick Ex:__ Find $e^t * \cos t$

Method I $\quad e^t * \cos t = \int_0^t e^{\theta} \cos(t-\theta)\, d\theta$

long integration by parts 11/7

Method II $\quad \mathcal{L}\{e^t * \cos t\} = \frac{1}{(s-1)} \frac{s}{s^2+1}$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$= \frac{1/2}{s-1} + \frac{-1/2 s + 1/2}{s^2+1} \quad \text{(check)}$$

$$e^t * \cos t = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

<u>Quick Ex:</u> Find $\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+4)}\right\}$ using a convolution.

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^2+4} \quad \frac{1}{s}\right\}$$

$$= f * g$$

$$\begin{cases} F(s) = \frac{2}{s^2+4} \\ G(s) = \frac{1}{s} \\ f(t) = \sin 2t \\ g(t) = 1 \end{cases}$$

$$= \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

$$= \int_0^t \sin 2\theta \; (1)\; d\theta$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^t$$

$$= -\frac{1}{2}\cos 2t + \frac{1}{2}$$

<u>Ex:</u> Solve for $f(t)$:

$$f(t) = 6t - \int_0^t f(\theta)\, e^{t-\theta}\, d\theta$$

$$f(t) = 6t - f * e^t$$

Apply $\mathcal{L}$:

$$F(s) = \frac{6}{s^2} - F(s)\,\frac{1}{s-1}$$

$$F(s) + \frac{1}{s-1}F(s) = \frac{6}{s^2}$$

$$\left[1 + \frac{1}{s-1}\right]F(s) = \frac{6}{s^2}$$

$$\frac{s}{s-1}F(s) = \frac{6}{s^2}$$

$$F(s) = \frac{6(s-1)}{s^3}$$

Apply $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6(s-1)}{s^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6\cancel{s}}{\cancel{s^3}\,s^2} - \frac{6}{s^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{6\,\frac{1}{s^2} - 3\,\frac{2!}{s^3}\right\}$$

$$= 6t - 3t^2$$

Table of Laplace Transforms

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$t \sin \omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$	$\dfrac{1}{(s^2+\omega^2)^2}$
$\delta(t)$	1

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{e^{at}t^n}{n!}$	$\dfrac{1}{(s-a)^{n+1}}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$t \cos \omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

$$\cancel{\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}}$$

7.3 $\quad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \cancel{\mathscr{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)}$

7.3 $\quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \qquad \mathscr{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$ $\quad$ 7.3

7.2 $\quad \mathscr{L}\{f'(t)\} = sF(s) - f(0) \qquad \mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ $\quad$ 7.2

$$\cancel{\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)}$$

7.4 $\quad f(t) * g(t) = \displaystyle\int_0^t f(\theta) g(t-\theta)\, d\theta \implies \mathscr{L}\{f(t) * g(t)\} = F(s)G(s)$

$$\cancel{\mathscr{L}\left\{\int_0^t f(\theta)\, d\theta\right\} = \frac{F(s)}{s}}$$

7.4 $\quad f(t)$ has period $T \implies \mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

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