7.2 Inverse Transforms

$\mathcal{L}\{t\} = \frac{1}{s^2}$ $\qquad$ $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

$\mathcal{L}\{f(t)\} = F(s)$ $\qquad$ $\mathcal{L}^{-1}\{F(s)\} = f(t)$

FACT

$\mathcal{L}^{-1}$ is linear:

$$\mathcal{L}^{-1}\{a F(s) + b\, G(s)\} = a \mathcal{L}^{-1}\{F(s)\} + b\, \mathcal{L}^{-1}\{G(s)\}$$

Ex: Find $\mathcal{L}^{-1}\{\frac{3}{s^2+16} - \frac{1}{s^3}\}$

$$= \mathcal{L}^{-1}\{\frac{3}{s^2+16}\} - \mathcal{L}^{-1}\{\frac{1}{s^3}\}$$

$$= \mathcal{L}^{-1}\{\left(\frac{4}{4}\right)\frac{3}{s^2+4^2}\} - \mathcal{L}^{-1}\{\frac{1}{2}\frac{2!}{s^3}\}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\{\frac{4}{s^2+4^2}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{2!}{s^3}\}$$

$$= \frac{3}{4}\sin 4t - \frac{1}{2}t^2$$

Ex: Find $\mathcal{L}^{-1}\{\frac{5s^2-5s+12}{s^3+s^2-12s}\}$

Partial Fractions $\qquad$ $s^3+s^2-12s = s(s^2+s-12)$

$$= s(s+4)(s-3)$$

$$\frac{5s^2-5s+12}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-3}$$

$$5s^2-5s+12 = A(s+4)(s-3) + Bs(s-3) + Cs(s+4)$$

$s=0: \quad 12 = A(-12) \Rightarrow A=-1$

$s=-4: \quad 112 = 28B \Rightarrow B=4$

$s=3: \quad 42 = 21C \Rightarrow C=2$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{4}{s+4} + \frac{2}{s-3}\right\}$$

$$= -1 + 4e^{-4t} + 2e^{3t}$$

<u>Ex</u>: Find $\mathcal{L}^{-1}\left\{\frac{12s^2+6}{s^4+s^2}\right\}$

Partial Fractions $\quad s^4+s^2 = s^2(s^2+1)$

repeated linear factor

quadratic factor

$$\frac{12s^2+6}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$12s^2+6 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

Can sub nice real or complex #

Can match coefficients

Can sub any real or complex #

$s=0: \quad 6=B$

Can sub $s=i$ because $i^2+1=0$

$s=i: \quad 12(-1)+6 = (Ci+D)(-1)$

$$\underbrace{-6}_{-6+0i} = -Ci-D$$

$D=6$

$C=0$

s^3 coefficient: $\quad 0 = A + C$

$A = 0$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \frac{6}{s^2} + \frac{6}{s^2+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ 6 \cdot \frac{1}{s^2} + 6 \cdot \frac{1}{s^2+1} \right\}$$

$$= 6t + 6\sin t$$

FACT

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Comment

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

<u>Ex</u>: Solve using $\mathcal{L}$:

$$y' + 7y = 65\cos 4t, \quad y(0) = 9$$

1) Apply $\mathcal{L}$

$$sY(s) - y(0) + 7Y(s) = \frac{65s}{s^2+16}$$

$$sY(s) - 9 + 7Y(s) = \frac{65s}{s^2+16}$$

2) Solve for $Y(s)$

$$sY(s) + 7Y(s) = 9 + \frac{65s}{s^2+16}$$

$$(s+7)Y(s) = 9 + \frac{65s}{s^2+16}$$

$$Y(s) = \frac{9}{s+7} + \frac{65s}{(s+7)(s^2+16)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{65s}{(s+7)(s^2+16)}\right\} \quad (*)$$

Partial Fractions

$$\frac{65s}{(s+7)(s^2+16)} = \frac{A}{s+7} + \frac{Bs+C}{s^2+16}$$

$$65s = A(s^2+16) + (Bs+C)(s+7)$$

$s=-7$: $\quad -455 = A(65) \implies A = -7$

s^2 coefficient: $\quad 0 = A + B \implies B = 7$

$s=0$: $\quad 0 = 16A + 7C \implies C = 16$

From $(*)$ $\quad y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s+16}{s^2+16}\right\}$

$$= 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s}{s^2+4^2} + 4\cdot\frac{4}{s^2+4^2}\right\}$$

$$= \underbrace{9e^{-7t} - 7e^{-7t}}_{2e^{-7t}} + 7\cos 4t + 4\sin 4t$$