2.2 Separable Variables

Ex: Solve $\frac{dy}{dx} = 2xy^2$

$$dy = 2xy^2\,dx$$

$$\frac{dy}{y^2} = 2x\,dx \qquad \text{Variables are separated "Separable DE"}$$

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$-y^{-1} = x^2 + C_1$$

Note: $-y^{-1} + C_1 = x^2 + C_2$ is unnecessary

$$-y^{-1} = x^2 + \underbrace{C_2 - C_1}_{C_3}$$

$$-\frac{1}{y} = x^2 + C_1$$

$$\frac{-1}{x^2+C_1} = y$$

$$y = \frac{-1}{x^2 + C}$$

Ex: Solve $\frac{dy}{dx} = \frac{y}{x}$

$$dy = \frac{y}{x}\,dx$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C_1$$

$e^{LS} = e^{RS}$:

$$e^{\ln|y|} = e^{\ln|x| + C_1}$$

$$|y| = e^{\ln|x|} \cdot e^{C_1}$$

$$|y| = |x| \cdot e^{C_1}$$

$$y = \pm x e^{C_1}$$

$$y = \underbrace{\left(\pm e^{C_1}\right)}_{C} x$$

$$y = Cx$$

Ex: Solve $\frac{dy}{dx} = \frac{y}{2-x}$ if $y(0) = 6$

Interval of solution?

$$dy = \frac{y}{2-x}\,dx$$

$$\frac{dy}{y} = \frac{dx}{2-x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{2-x}$$

$u = 2-x$

$du = -dx$

$-du = dx$

$\int \frac{-du}{u}$

$= -\ln|u| + C_1$

$= -\ln|2-x| + C_1$

$$\ln|y| = -\ln|2-x| + C_1$$

$$e^{\ln|y|} = e^{-\ln|2-x| + C_1}$$

$$|y| = e^{-\ln|2-x|} \cdot e^{C_1}$$

$$|y| = e^{\ln|2-x|^{-1}} \cdot e^{C_1}$$

$$a \ln b = \ln b^a$$

$$|y| = |2-x|^{-1} \cdot e^{C_1}$$

$$y = \pm \frac{e^{C_1}}{2-x}$$

$$y = \frac{C}{2-x}$$

$y=6$, $x=0$:

$$6 = \frac{C}{2}$$

$$C = 12$$

$$y = \frac{12}{2-x}$$

Interval: $x \neq 2$

$x < 2$ or $x > 2$

But $x=0$ was given

$$\boxed{x<2}$$

<u>Ex</u>: Solve $x\,dx + y\,dy = 0$ with $y(2) = -1$

$$x\,dx = -y\,dy$$

$$\int x\,dx = \int -y\,dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + C_1$$

$$x^2 = -y^2 + \cancel{2C_1}\ C_2$$

$y=-1$, $x=2$:

$$4 = -1 + C_2$$

$$C_2 = 5$$

$$x^2 = -y^2 + 5$$

$$y^2 = 5 - x^2$$

$$y = \pm\sqrt{5-x^2}$$

But $y(2) = -1 \Rightarrow$ Choose $\ominus$

$$y = -\sqrt{5-x^2}$$

<u>Ex:</u> Solve $\dfrac{dy}{dx} = y^2 - 9$

$$dy = (y^2-9)\,dx$$

$$\frac{dy}{y^2-9} = dx$$

$$\int \frac{dy}{y^2-9} = \int dx$$

Partial Fractions

$$\frac{1}{y^2-9} = \frac{1}{(y-3)(y+3)}$$

$$\frac{1}{(y-3)(y+3)} = \frac{A}{y-3} + \frac{B}{y+3}$$

$$1 = A(y+3) + B(y-3)$$

$y=-3$: $1 = B(-6) \Rightarrow B = \frac{-1}{6}$

$y=3$: $1 = A(6) \Rightarrow A = \frac{1}{6}$

$$\int \left[\frac{1}{6}\frac{1}{y-3} - \frac{1}{6}\frac{1}{y+3}\right]dy = \int dx$$

$$\int \frac{1}{x}dx = \ln|x| + C_1$$

$$\int \frac{1}{x+k}dx = \ln|x+k| + C_1$$

$$\int \frac{1}{ax+k}dx = \frac{1}{a}\ln|ax+k| + C_1$$

$$\frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y+3| = x + C_1$$

$$\ln|y-3| - \ln|y+3| = 6x + C_2$$

Algebra To be Continued ...

Detour: Trig Sub (Just for Practice)

$x^2 - a^2 \Rightarrow \quad x = a\sec\theta$

$y^2 - 9 \Rightarrow \quad y = 3\sec\theta$

$dy = 3\sec\theta\tan\theta\, d\theta$

$$\int \frac{dy}{y^2-9} = \int \frac{3\sec\theta\tan\theta\, d\theta}{(9\sec^2\theta - 9)}$$

$$= \frac{1}{3}\int \frac{\sec\theta\tan\theta\, d\theta}{\tan^2\theta}$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}d\theta$$

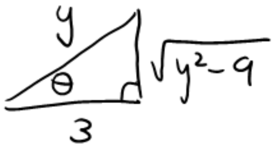
$$\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$= \frac{1}{3}\int \csc\theta\, d\theta$$

$$= \frac{1}{3}\ln|\csc\theta - \cot\theta| + C_1$$

$y = 3\sec\theta$

$\frac{y}{3} = \sec\theta = \frac{H}{A}$



$$9 + ?^2 = y^2$$

$$? = \sqrt{y^2-9}$$

$$= \frac{1}{3}\ln\left|\frac{y}{\sqrt{y^2-9}} - \frac{3}{\sqrt{y^2-9}}\right| + C_1$$

$$\vdots$$

$$= \frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y+3| + C_1$$

Partial Fractions is usually easier than Trig Sub!