9.10 Cont'd (Taylor Series)

Ex: Find the Maclaurin series for $x^3 \sin x^2$, given $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$x^3 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}$$

Ex: Find the first 3 nonzero terms of the Maclaurin series for $e^x \sqrt{1+x}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2} x^2 + \dots$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{1}{2}\left(\frac{-1}{2}\right)\frac{x^2}{2} + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$$

$$e^x \sqrt{1+x} = \left(1 + x + \frac{x^2}{2} + \dots\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + \dots\right)$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8}$$

$$x + \frac{x^2}{2}$$

$$\frac{x^2}{2}$$

$$= 1 + \frac{3x}{2} + \frac{7x^2}{8} + \dots$$

Ex: a) Approximate $\int_0^{0.5} e^{-x^2}\,dx$

using 4 nonzero terms.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{6} + \dots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

$$\int_0^{0.5} e^{-x^2}\,dx \approx \int_0^{0.5} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx$$

$$\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42}$$

$$\approx 0.46127232$$

b) Find an upper bound for $|\text{error}|$

$0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} + \ldots$ is an alternating series

$$|R_N| \le a_{N+1}$$

$\Leftarrow$ absolute value of next term

Next term of e^x: $\frac{x^4}{24}$

e^{-x^2}: $\frac{x^8}{24}$

$\int_0^{0.5} e^{-x^2}\,dx$: $\frac{0.5^9}{24(9)}$

$$|\text{error}| \le \frac{0.5^9}{24(9)}$$

$$\le 9.0 \times 10^{-6}$$

c) Estimate $\int_0^{0.5} e^{-x^2}\,dx$

$$\int_0^{0.5} e^{-x^2}\,dx = 0.46127232 \pm 9.0 \times 10^{-6}$$

d) How many nonzero terms are required for $|\text{error}| \le 0.01$?

$$\int_0^{0.5} e^{-x^2}\,dx \approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \ldots$$

(Find the first term that is ≤ 0.01)

$\Leftarrow$ this is ≤ 0.01 (in absolute value)

2 terms needed

9.10 #47

Find the first 4 nonzero terms of $e^x \sin x$

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right)$$

$$= \quad x \quad - \frac{x^3}{6} \quad + \frac{x^5}{120}$$

$$\quad + x^2 \quad - \frac{x^4}{6}$$

$$\quad + \frac{x^3}{2} \quad - \frac{x^5}{12}$$

$$\quad + \frac{x^4}{6}$$

$$\quad + \frac{x^5}{24}$$

$$= \quad x + x^2 + \frac{x^3}{3} \quad - \frac{x^5}{30} + \ldots$$