9.9 Power Series Representations Cont'd

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$

<u>Ex</u>: Find a power series for $\arctan x$ centred at $c=0$.

Aside

$\sum b_n (x-c)^n$ centred at c

$\sum b_n x^n$ " 0

Notice $\arctan x + C_1 = \int \frac{1}{1+x^2}\,dx$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan x + C_1 = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$x=0$: $\quad 0 + C_1 = 0$

$$C_1 = 0$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Ex: a) Approximate $\int_0^{0.5} \frac{1}{1+x^5}\,dx$ using 3 terms of a power series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - \ldots$$

$$\frac{1}{1+x^5} = 1 - x^5 + x^{10} - \ldots$$

$$\int_0^{0.5} \frac{1}{1+x^5}\,dx \approx \int_0^{0.5} (1 - x^5 + x^{10})\,dx$$

$$\approx \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} \right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^6}{6} + \frac{0.5^{11}}{11}$$

$$\approx 0.49744022$$

b) Find an upper bound for |error|

$0.5 - \frac{0.5^6}{6} + \frac{0.5^{11}}{11} - \ldots$ is an alternating series

$$|R_N| \leq a_{N+1}$$

absolute value of next term

$$|\text{error}| \le \frac{0.5^{16}}{16} \le 9.6 \times 10^{-7}$$

c) Estimate the integral

$$0.49744022 - 9.6\times10^{-7} \le \int_0^{0.5} \frac{1}{1+x^5}\,dx \le 0.49744022 + 9.6\times10^{-7}$$

9.9 #11

Find a power series for $f(x) = \frac{5}{2x-3}$ centred at $c = -3$ and find the interval of convergence.

$$\frac{5}{2x-3} = \frac{5}{2(x+3)+?}$$

$$= \frac{5}{2(x+3)-9}$$

$$= \frac{5}{-9+2(x+3)}$$

$$= \frac{-5}{9}\left[\frac{1}{1-\frac{2}{9}(x+3)}\right]$$

$$= -\frac{5}{9}\sum_{n=0}^{\infty} \underbrace{\left[\frac{2}{9}(x+3)\right]^n}_{a_n}$$

Interval of Convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left[\frac{2}{9}(x+3)\right]^{n+1}}{\left[\frac{2}{9}(x+3)\right]^n}\right|$$

$$= \left|\frac{2}{9}(x+3)\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2}{9}(x+3)\right|$$

Series converges if $\left|\frac{2}{9}(x+3)\right| < 1$

$$|x+3| < \frac{9}{2}$$

$$-\frac{9}{2} < x+3 < \frac{9}{2}$$

$$-\frac{15}{2} < x < \frac{3}{2}$$

$x = -\frac{15}{2}$:

$$\sum_{n=0}^{\infty}\left[\frac{2}{9}\left(-\frac{9}{2}\right)\right]^n$$

$$= \sum_{n=0}^{\infty}(-1)^n$$

Diverges

n^{th} term test

$x = \frac{3}{2}$:

$$\sum_{n=0}^{\infty}\left[\frac{2}{9}\left(\frac{9}{2}\right)\right]^n$$

$$= \sum_{n=0}^{\infty}1^n$$

Diverges

n^{th} term test

$$\boxed{-\frac{15}{2} < x < \frac{3}{2}}$$

9.10 Taylor and Maclaurin Series

The Taylor series of f centred at c:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ it's called the Maclaurin series of f.

Ex: Find the Maclaurin series of $f(x)=\sin x$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

Comments

- Interval of convergence for above series is $-\infty < x < \infty$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$ (not an approximation)
- If a question asks you to find a series, you may give the first 3 nonzero terms.

Ex: Find the Maclaurin series of $f(x) = (1+x)^k$, where k is a real #.

$f(x) = (1+x)^k \qquad f(0) = 1$

$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0) = k(k-1)$

$$(1+x)^k = \frac{1}{0!}x^0 + \frac{k}{1!}x^1 + \frac{k(k-1)}{2!}x^2 + \ldots$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$$