# 9.8 Cont'd

Ex: Find the interval of convergence:

$$\sum_{n=1}^{\infty} \underbrace{\frac{(x-4)^n}{n \cdot 9^n}}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}} \cdot \frac{n\, 9^n}{(x-4)^n}\right|$$

$$= \left|\frac{(x-4)}{9} \frac{n}{n+1}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-4}{9}(1)\right|$$ L'Hôpital's Rule

$$= \left|\frac{x-4}{9}\right|$$

Series converges if $\left|\frac{x-4}{9}\right| < 1$

$$|x-4| < 9$$

$$-9 < x-4 < 9$$

$$-5 < x < 13$$

$x=-5$:

$$\text{Series} = \sum_{n=1}^{\infty} \frac{(-9)^n}{n \cdot 9^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges (Alternating)

$x=13$:

$$\text{Series} = \sum_{n=1}^{\infty} \frac{9^n}{n \cdot 9^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges (p-series)

$$\boxed{-5 \le x < 13}$$

**Ex:** Find the interval of convergence:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \left|\frac{x}{n+1}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$$

The series converges for all $x$.

Interval of Convergence: $-\infty < x < \infty$

Radius " : $\infty$

**Ex:** Find the interval of convergence:

$$\sum_{n=0}^{\infty} n!\,(x-2)^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}\right|$$

$$= \left|(n+1)\,(x-2)\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \infty & , x \neq 2 \\ 0 & , x = 2 \end{cases}$$

The series Converges at $x=2$

Interval of Convergence: $x=2$

Radius " : $0$

FACT

Suppose $f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$

has radius of convergence $R > 0$.

Then $f'(x)$ and $\int f(x)dx$ also have

radius of convergence $= R$.

$$f'(x) = \sum_{n=0}^{\infty} n\, b_n (x-c)^{n-1} \qquad (*)$$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{b_n (x-c)^{n+1}}{n+1} + C_1 \qquad (*)$$

on the interval $c-R < x < c+R$

Ex: Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ has $R=1$.

Find $f'(x)$ and $\int f(x)dx$ on

the interval $-1<x<1$:

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1}$$

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} + C$$

# 9.9 Power Series Representations

Recall $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1 < r < 1$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and $\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$

FACT

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

valid for $-1 < x < 1$

<u>Ex</u>: Find a power series for $\frac{1}{1+x^3}$ centred at $c = 0$.

Centred at $0$ : $\sum b_n x^n$

Centred at $c$ : $\sum b_n (x-c)^n$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

<u>Ex</u>: Find a power series for $\frac{10}{3+2x}$ centred at $c=1$.

$$\frac{10}{3+2x} = \frac{10}{?+2(x-1)}$$

$$= \frac{10}{5+2(x-1)}$$

$$= \frac{10}{5}\left[\frac{1}{1+\frac{2}{5}(x-1)}\right]$$

$$= 2\sum_{n=0}^{\infty}\left[-\frac{2}{5}(x-1)\right]^n$$

$$= 2\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n (x-1)^n$$

Ex: Find a power series for

$\frac{1}{(1-x)^2}$ centred at $c=0$.

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}$$

Check: $\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$ ✓

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

Ex: Find a power series for

$\ln(1+x)$ centred at $c=0$.

$$\ln(1+x) + C_1 = \int \frac{1}{1+x}\, dx$$

Check: $\int \frac{1}{1+x}\, dx = \ln|1+x| + C_1$

We'll assume $1+x > 0$ ✓

$$\ln(1+x) + C_1 = \int \frac{1}{1+x}\, dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\, dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$x = 0$: $\cancel{\ln 1}_{0} + C_1 = 0$

$$C_1 = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$