9.7 Taylor and Maclaurin Polynomials

The <u>n^{th} degree Taylor polynomial of $f(x)$ centred at c</u> is:

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

$$+ \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ then it's called the <u>n^{th} degree Maclaurin polynomial</u>.

<u>Ex</u>: Find the 7^{th} degree Maclaurin polynomial of $f(x) = \sin x$.

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -1$

$$P_7(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(7)}(0)}{7!}x^7$$

$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \checkmark$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \checkmark$$

Ex: Find the n^{th} degree Taylor polynomial of $f(x) = \ln x$ centred at $c = 1$.

$f(x) = \ln x$ $\quad$ $f(1) = 0$

$f'(x) = x^{-1}$ $\quad$ $f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad$ $f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\quad$ $f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad$ $f^{(4)}(1) = -6$

$\vdots$

$f^{(5)}(1) = 24$

$f^{(6)}(1) = -120$

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

$$P_n(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots$$

$$+ \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \dots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \dots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder or error}}$$

FACT

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

where z is a number between x and c.

Ex: a) Find the 2^{nd} degree Taylor polynomial of $f(x) = \sqrt{x}$ centred at $c = 4$.

$f(x) = x^{1/2}$ $\qquad$ $f(4) = 2$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad$ $f'(4) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$f''(x) = \frac{-1}{4} x^{-3/2}$ $\qquad$ $f''(4) = \frac{-1}{4}\left(\frac{1}{8}\right) = \frac{-1}{32}$

$$P_2(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

b) Approximate $\sqrt{4.1}$

$$P_2(4.1) = 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2$$

$$\approx 2.02484375$$

c) Find an upper bound for $|R_2(4.1)|$

Recall $\quad f''(x) = -\frac{1}{4} x^{-3/2}$

$f'''(x) = \frac{3}{8} x^{-5/2}$

$f'''(z) = \frac{3}{8} z^{-5/2}$

$$R_2(x) = \frac{f'''(z)}{3!}(x-4)^3$$

$$= \frac{1}{6}\cdot\frac{3}{8} z^{-5/2}(x-4)^3$$

$$|R_2(4.1)| = \frac{1}{6}\cdot\frac{3}{8} z^{-5/2}(0.1)^3$$

where z is between 4 and 4.1

$$|R_2(4.1)| \le \frac{1}{6}\cdot\frac{3}{8}(4)^{-5/2}(0.1)^3$$

$$\le 0.000002$$

d) Approximate $\sqrt{4.1}$, with error

$$2.02484375 - 0.000002 \le \sqrt{4.1} \le 2.02484375 + 0.000002$$

9.7 #21

Find the 4th degree Maclaurin polynomial of $f(x) = xe^x$

$$f(x) = xe^x \qquad f(0) = 0$$

$$f'(x) = xe^x + e^x \qquad f'(0) = 1$$

$$f''(x) = (xe^x + e^x) + e^x \qquad f''(0) = 2$$

$$= xe^x + 2e^x$$

$$f'''(x) = xe^x + e^x + 2e^x$$
$$= xe^x + 3e^x$$
$$f^{(4)}(x) = xe^x + 4e^x$$

$$f'''(0) = 3$$
$$f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(4)}(0)}{4!}x^4$$

$$= x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

<u>Ex</u>: Find the 1^{st} degree Taylor polynomial of $f(x) = \frac{\sqrt{x}}{4}$ centred at $c=4$.

$$f(x) = \frac{1}{4}x^{1/2}$$
$$f'(x) = \frac{1}{8}x^{-1/2}$$

$$f(4) = \frac{1}{4}(2) = \frac{1}{2}$$
$$f'(4) = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16}$$

$$P_1(x) = f(4) + \frac{f'(4)}{1!}(x-4)$$

$$= \frac{1}{2} + \frac{1}{16}(x-4)$$