9.5 Cont'd

Alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

In an alternating series, a_n is unsigned.

e.g. $a_n = \frac{1}{n}$

Recall:

If a series converges by the Alternating Series Test then

$$|R_N| \le a_{N+1}$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+1}$ converges by the Alternating Series Test. Find N so that $|R_N| \le 0.005$

Plan: $a_{N+1} \le 0.005$

(Then $|R_N| \le a_{N+1} \le 0.005$)

$$a_n = \frac{1}{\sqrt{n}+1}$$

$$a_{N+1} = \frac{1}{\sqrt{N+1}+1}$$

$$\frac{1}{\sqrt{N+1}+1} \le 0.005$$

$$\cancel{\frac{1}{0.005}}\ 200 \le \sqrt{N+1}+1$$

$$199 \le \sqrt{N+1}$$

$$39601 \le N+1$$

$$39600 \le N$$

$$\boxed{N \geq 39600}$$

Def

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating). $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u> if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ex: a) $1 - \frac{1}{2} + \frac{1}{3} - \dots$ Converges (Alternating)

$1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges (p-series)

$1 - \frac{1}{2} + \frac{1}{3} - \dots$ is conditionally convergent

b) $1 - \frac{1}{4} + \frac{1}{9} - \dots$ Converges (Alternating)

$1 + \frac{1}{4} + \frac{1}{9} + \dots$ Converges (p-series)

$1 - \frac{1}{4} + \frac{1}{9} - \dots$ is absolutely convergent

Absolute Convergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating). If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges.

Is it conditionally convergent or absolutely convergent?

Analyze $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^3}$.

Use Direct Comparison Test.

$$0 < \frac{|\cos n|}{n^3} \leq \frac{1}{n^3} \quad \text{for } n \geq 1 \checkmark$$

(Note: $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\cos(\text{integer}) \neq 0$)

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} \frac{|\cos n|}{n^3}$ Converges

$\boxed{\text{Original series Converges absolutely.}}$

9.6 The Ratio and Root Tests

Ratio Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $L < 1$ then the series converges absolutely.

$L = 1$ then the test is inconclusive.

$L > 1$ then the series diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{3^n} \longleftarrow \quad a_n = \frac{(-1)^{n+1} n^3}{3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3\,\cancel{3^{n+1}}} \cdot \frac{\cancel{3^n}}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$= \frac{1}{3}$$

The series converges absolutely.

Recall: Section 5.6

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \longleftarrow \quad a_n = \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \frac{(n+1)^{n+1}}{n^n} \cdot \cancel{\frac{n!}{(n+1)!}} \ \frac{1}{(n+1)}$$

$$= \frac{\cancel{(n+1)}(n+1)^n}{n^n} \cdot \frac{1}{\cancel{n+1}}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e$$

$\boxed{\text{The series diverges.}}$

Root Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

If $L<1$ then the series converges absolutely.

$L=1$ no info

$L>1$ then the series diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+5}\right)^n \longleftarrow a_n = \left(\frac{2n+3}{3n+5}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+5}\right)^n}$$

$$= \lim_{n\to\infty}\frac{2n+3}{3n+5} \longleftarrow \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty}\frac{2}{3}$$

$$= \frac{2}{3}$$

The series converges absolutely.