9.4 Comparisons of Series

Direct Comparison Test

Suppose $0 < a_n \leq b_n$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Also true if index starts at $n=0$ etc.

Recall: $\sum_{n=0}^{\infty} ar^n$ converges if $-1 < r < 1$
diverges if $|r| \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \leq 1$

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{1 + \ln n}{n}$$

$$\frac{1+\ln n}{n} \geq \frac{1}{n} \quad \text{for all } n$$

$$0 < \frac{1}{n} \leq \frac{1+\ln n}{n} \quad \text{for all } n \checkmark$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1+\ln n}{n} \quad \text{diverges}$$

Ex: Test for convergence:

$$\sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$$

$$n^2 \geq n \quad \text{for all } n$$

$$2^{(n^2)} \geq 2^n \quad \text{"}$$

$$\frac{1}{2^{(n^2)}} \leq \frac{1}{2^n} \quad \text{"}$$

$$0 < \frac{1}{2^{(n^2)}} \leq \frac{1}{2^n} \qquad \text{for all } n \checkmark$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \text{ converges } \left(\text{geometric } r = \tfrac{1}{2}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}} \text{ converges}$$

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge or both diverge.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{1}{3n+5}$$

Look at dominant terms: $\frac{1}{n}$

$\frac{1}{3n+5} > 0$ and $\frac{1}{n} > 0$ for all n ✓

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{3n+5}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n}{3n+5}$$

$$\overset{H}{=} \frac{1}{3}$$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n+5}$ diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{3\sqrt{n}+2}{5n^2+3n+1}$$

Look at dominant terms: $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$\frac{3\sqrt{n}+2}{5n^2+3n+1} > 0$ and $\frac{1}{n^{3/2}} > 0$ for all n ✓

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{3\sqrt{n}+2}{5n^2+3n+1}\right)}{\left(\frac{1}{n^{3/2}}\right)}$$

$$= \lim_{n\to\infty} \frac{(3\sqrt{n}+2)}{5n^2+3n+1} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{3n^2 + 2n^{3/2}}{5n^2+3n+1}$$

$$= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} + \frac{2n^{3/2}}{n^2}}{\frac{5n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3 + \frac{2}{n^{1/2}}}{5 + \frac{3}{n} + \frac{1}{n^2}}$$

$$= \frac{3}{5}$$ (or do L'Hôpital's TWICE)

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{3\sqrt{n}+2}{5n^2+3n+1}$ Converges

Direct Comparison Test works well when n is in the exponent or when ln n appears in the term.

Limit Comparison Test works well for ratios of polynomials.

Extra: Test for Convergence

a) $\sum_{n=1}^{\infty} \frac{1+|\sin n|}{\sqrt{n}}$

Direct Comparison

$$\frac{1+|\sin n|}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \quad \text{for all } n$$

$$0 < \frac{1}{\sqrt{n}} \leq \frac{1+|\sin n|}{\sqrt{n}} \quad \text{for all } n \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

$\Rightarrow$ Series diverges

b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+1}}$

Limit Comparison Test

Look at dominant terms: $\frac{1}{\sqrt{n^2}} = \frac{1}{n}$

$$\frac{2}{\sqrt{n^2+1}} > 0 \quad \text{and} \quad \frac{1}{n} > 0 \quad \text{for all } n \checkmark$$

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2}{\sqrt{n^2+1}}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} 2\cdot\sqrt{\frac{n^2}{n^2+1}}$$

$$= \lim_{n\to\infty} 2\cdot\sqrt{\frac{1}{1+\frac{1}{n^2}}}$$

$$= 2 \quad \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

$\Rightarrow$ Series diverges.