9.3 The Integral Test and p-Series

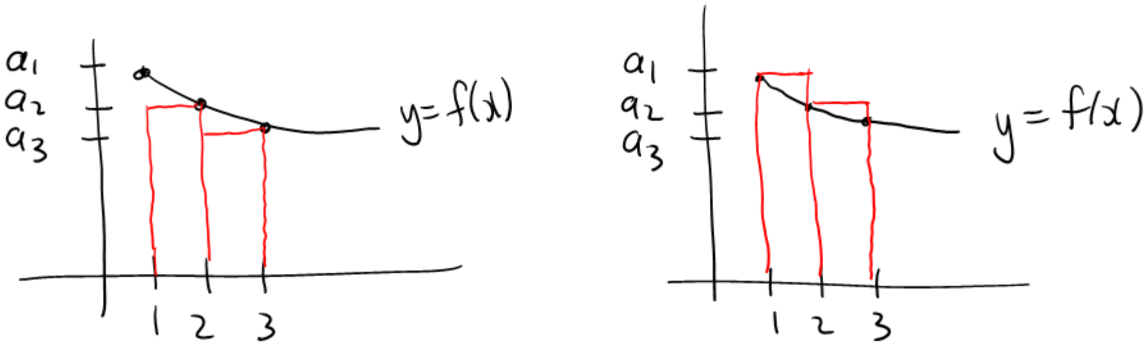
The Integral Test

If f is continuous, positive and decreasing on $[1,\infty)$ and $a_n = f(n)$ for $n = 1, 2, 3, \ldots$

then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$

both converge or both diverge.

why?



$$a_2 + a_3 + \ldots \leq \int_1^{\infty} f(x)\,dx \leq a_1 + a_2 + \ldots$$

$\sum_{n=1}^{\infty} a_n$ is a finite number

$\Longleftrightarrow$ $\int_1^{\infty} f(x)\,dx$ is a finite number

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

$f(x) = \frac{1}{x^2}$

$f(x)$ is continuous on $[1, \infty)$ ✓

$f(x)$ is positive " ✓

$f'(x) = -2x^{-3}$

$f'(x) < 0$ on $[1, \infty)$

$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

$$= \lim_{b \to \infty} \int_1^{b} \frac{1}{x^2}\, dx$$

$$= \lim_{b \to \infty} \left. \frac{-1}{x} \right|_1^b$$

$$= \lim_{b \to \infty} \frac{-1}{b} + 1$$

$$= 1$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

CAUTION: No info about the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

$f(x) = \frac{1}{x}$

$f(x)$ continuous on $[1, \infty)$ ✓

$f(x)$ positive " ✓

$f'(x) = \frac{-1}{x^2} < 0$ on $[1, \infty)$ ✓

$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b - \ln 1$$

$$= \infty$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

FACT

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$

Terminology: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a <u>p-series</u>.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is called the <u>harmonic series</u>

Recap of Section 9.2 / 9.3

GEOMETRIC $\sum_{n=1}^{\infty} 5(0.2)^n = 1.25$

$\sum_{n=1}^{\infty} 5(1.2)^n = \infty$ (diverges)

TELESCOPING $\sum_{n=1}^{\infty} \left(\frac{3}{n} - \frac{3}{n+1}\right) = 3$

---

n^{th} term test $\sum_{n=1}^{\infty} n$ DIVERGES

because $\lim_{n\to\infty} n \neq 0$

Integral Test $\sum_{n=1}^{\infty} \frac{1}{n^3}$ CONVERGES