9.2 Series and Convergence

Sequence: $a_1, a_2, a_3, \dots$ (LIST)

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ (SUM)

The N^{th} partial sum of a series is the sum of the first N terms. Written S_N.

Ex: Find the partial sums S_1, S_2, S_3.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

Notation: Let $S = \lim_{N\to\infty} S_N$ (if it exists)

Def

If $S = \lim_{N\to\infty} S_N$ exists and is a real number then the series converges. Otherwise it diverges.

A geometric series is:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

FACT

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if} \quad -1<r<1$$

$$\sum_{n=0}^{\infty} ar^n \text{ diverges} \quad \text{if} \quad |r| \geq 1$$

<u>Ex</u>: Let $-1<r<1$. Consider $\sum_{n=0}^{\infty} ar^n$.
Show $S = \frac{a}{1-r}$

$$S_N = a + ar + \dots + ar^{N-1}$$

$$rS_N = ar + ar^2 + \dots + ar^N$$

$$S_N - rS_N = a - ar^N$$

$$S_N(1-r) = a - ar^N$$

$$S_N = \frac{a-ar^N}{1-r}$$

$$S = \lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty} \frac{a - ar^N}{1-r} \qquad ar^N \to 0$$

$$= \frac{a}{1-r}$$

Ex: Find the sum or state that it diverges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n} = 3 + \frac{3}{4} + \frac{3}{16} + \dots$

geometric series $a = 3$ $r = \frac{1}{4}$ "ratio"

$$S = \frac{a}{1-r}$$

$$= \frac{3}{\left(\frac{3}{4}\right)}$$

$$= 3 \times \frac{4}{3}$$

$$= 4$$

Series converges to 4.

b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \dots$

geometric series $r = \frac{3}{2}$

Series diverges.

c) $\displaystyle\sum_{n=2}^{\infty} \frac{7(3^{n-1})}{5^n} = \frac{21}{25} + \frac{63}{125} + \dots$

geometric series $a = \frac{21}{25}$ $r = \frac{3}{5}$

$$S = \frac{a}{1-r}$$

$$= \frac{\left(\frac{21}{25}\right)}{\left(\frac{2}{5}\right)}$$

$$= \frac{21}{25} \times \frac{5}{2}$$

$$= \frac{21}{10}$$

d) $\sum_{n=k}^{\infty} ar^n$ where $-1<r<1$

geometric series 1^{st} term $= ar^k$ ratio $= r$

$$S = \frac{1^{st}\text{ term}}{1-\text{ratio}}$$

$$= \frac{ar^k}{1-r}$$

<u>Ex</u>: Write $0.\overline{4}$ as a fraction

$$0.\overline{4} = 0.444\ldots$$

$$= 0.4 + 0.04 + 0.004 + \ldots$$

$\boxed{\text{geometric series} \quad a = 0.4 \quad r = 0.1}$

$$= \frac{a}{1-r}$$

$$= \frac{0.4}{0.9} \times \frac{10}{10}$$

$$= \frac{4}{9}$$

FACT

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge

then $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$ (c: any real #).

<u>Ex</u>: Find $\sum_{n=2}^{\infty} \frac{12 + 4(2^{n+1})}{5^n}$

$$= \underbrace{\sum_{n=2}^{\infty} \frac{12}{5^n}}_{\substack{\text{geometric} \\ a = \frac{12}{25} \; r = \frac{1}{5}}} + \underbrace{\sum_{n=2}^{\infty} \frac{4(2^{n+1})}{5^n}}_{\substack{\text{geometric} \\ a = \frac{32}{25} \; r = \frac{2}{5}}}$$

$$= \frac{\left(\frac{12}{25}\right)}{\left(\frac{4}{5}\right)} + \frac{\left(\frac{32}{25}\right)}{\left(\frac{3}{5}\right)}$$

$$= \frac{12}{25} \times \frac{5}{4} + \frac{32}{25} \times \frac{5}{3}$$

$$= \frac{3}{5} + \frac{32}{15}$$

$$= \frac{41}{15}$$

A series of the form

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + \ldots$$

is a <u>telescoping series</u>.

Partial Sums:

$$S_1 = b_1 - b_2$$
$$S_2 = b_1 \cancel{- b_2 + b_2} - b_3 = b_1 - b_3$$
$$S_3 = b_1 - b_4$$
$$S_N = b_1 - b_{N+1}$$

FACT

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{N\to\infty} b_N$$

if the limit exists.

Ex: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Use partial fractions to write as a telescoping series.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

Sub $n=0$: $1 = A$

$n=-1$: $1 = B(-1) \Rightarrow B = -1$

$$S = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

$$= b_1 - \lim_{N\to\infty} b_N$$

$$\boxed{b_n = \frac{1}{n}}$$

$$= 1 - \lim_{N\to\infty} \frac{1}{N}$$

$$= 1$$

FACT : n^{th} Term Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: Consider $\sum_{n=1}^{\infty} \frac{3n+1}{5n+1}$

$$\lim_{n\to\infty} \frac{3n+1}{5n+1} \leftarrow \frac{\infty}{\infty}$$

$$\overset{H}{=} \frac{3}{5}$$

The series diverges.

Caution: If $\lim_{n\to\infty} a_n = 0$

the series may or may not converge.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(details later)

but both have $\lim_{n\to\infty} a_n = 0$.