8.8 Improper Integrals

See handout on website. Read first page.

<u>Ex</u>: Evaluate or show that it diverges.

a) $$\int_1^\infty \frac{1}{x}\,dx$$

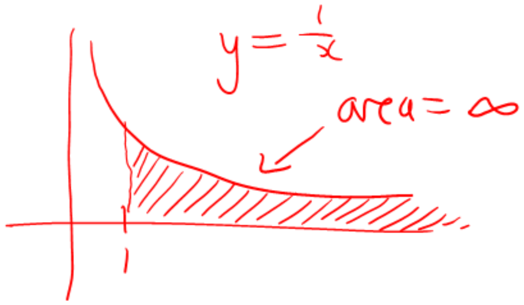
$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \ln b$$

$$= \infty$$

The integral diverges.



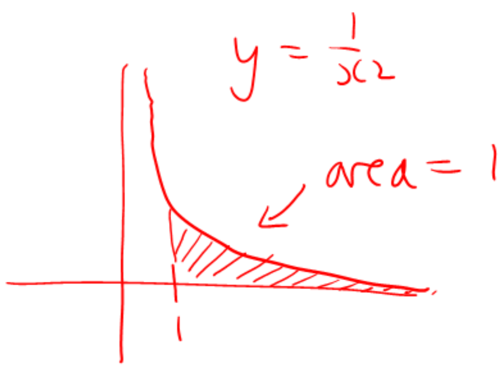
b) $$\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \Big[-x^{-1}\Big]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b}+1\right)$$

$$= 1$$



FACT

Let p be a real number.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & , \text{ if } p > 1 \\ \infty & , \text{ if } p \le 1 \end{cases}$$

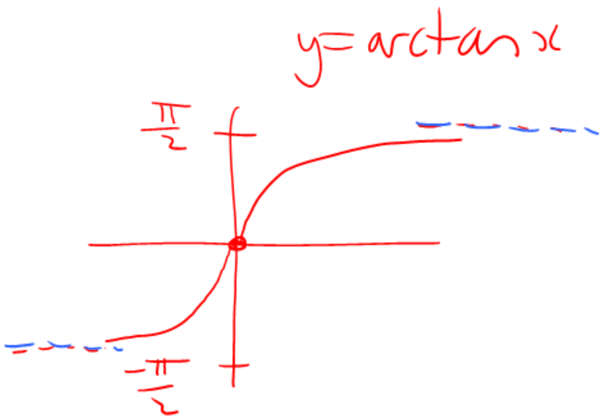
<u>Ex</u>: a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

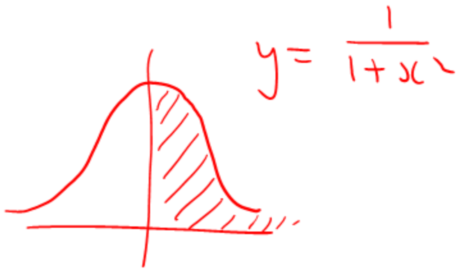
$$= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\arctan x\right]_0^b$$

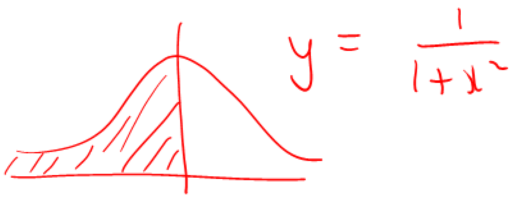
$$= \lim_{b\to\infty} \arctan b$$

$$= \frac{\pi}{2}$$





b) $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$



$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx$$

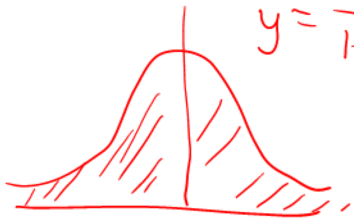
$$= \lim_{a \to -\infty} \left[\arctan x\right]_a^0$$

$$= \lim_{a \to -\infty} 0 - \arctan a$$

$$= \quad - \left(-\frac{\pi}{2}\right)$$

$$= \quad \frac{\pi}{2}$$

c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

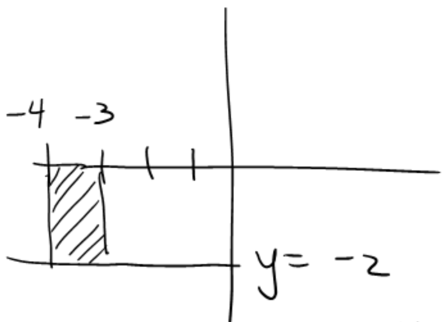
ASIDE :

$$\int_{-4}^{-3} (-2)\,dx$$

$$= -2x \Big|_{-4}^{-3}$$

$$= 6 - 8$$

$$= -2$$



If $f(x) < 0$ then $\int_a^b f(x)\,dx$ may be < 0.

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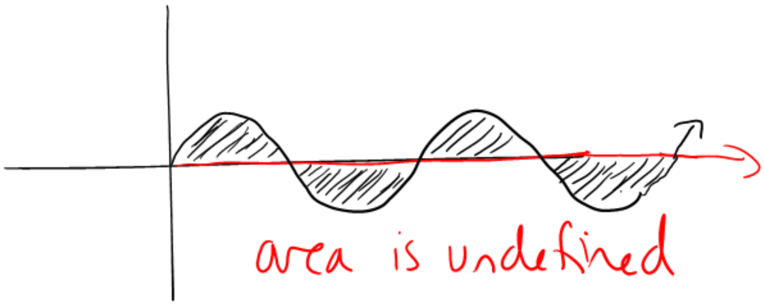
d) $\displaystyle\int_0^\infty \sin x \, dx$

$$= \lim_{b\to\infty} \int_0^b \sin x \, dx$$

$$= \lim_{b\to\infty} \left[-\cos x\right]_0^b$$

$$= \lim_{b\to\infty} -\cos b + 1$$

$=$ undefined (cos b oscillates as $b \to \infty$)



The integral diverges.

e) $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx$

$$= \int_{-\infty}^{0} \sin x \, dx + \underbrace{\int_0^\infty \sin x \, dx}_{\text{diverges}}$$

Therefore $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx$ diverges.