5.6 Indeterminate Forms

Indeterminate forms:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0(\infty)$, $\infty - \infty$, 0^0, ∞^0, 1^∞

The limit may or may not exist.

L'Hôpital's Rule

Suppose $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

as long as the second limit exists (or is $\pm\infty$).

<u>Note</u>: $x \to a$ also includes

$x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$

<u>Ex</u>: $\lim_{x \to \infty} \frac{x^2+1}{3x^2+5x+4}$

The form is $\frac{\infty}{\infty}$ ✓

$$\lim_{x \to \infty} \frac{x^2+1}{3x^2+5x+4} \overset{H}{=} \lim_{x \to \infty} \frac{2x}{6x+5}$$

$\frac{\infty}{\infty}$ ✓

$$\overset{H}{=} \lim_{x \to \infty} \frac{2}{6}$$

$$= \frac{1}{3}$$

The form must be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$ each time l'Hôpital's Rule is applied.

Ex: $\lim\limits_{x \to 0} \frac{\sin x}{x}$

$\frac{0}{0}$ ✓

$\overset{\text{H}}{=} \lim\limits_{x \to 0} \frac{\cos x}{1}$

$= 1$

Ex: $\lim\limits_{x \to 2} \frac{1-\cos(x^3-8)}{(x-2)^2}$

$\frac{0}{0}$ ✓

$\overset{\text{H}}{=} \lim\limits_{x \to 2} \frac{3x^2 \sin(x^3-8)}{2(x-2)}$ ⟵ $\frac{0}{0}$ ✓

$$\overset{H}{=} \lim_{x\to 2} \frac{9x^4 \cos(x^3-8) + 6x\sin(x^3-8)}{2}$$

$$= \lim_{x\to 2} \frac{144}{2}$$

$$= 72$$

$\underline{Ex}$: $$\lim_{x\to\infty} \frac{3x^2+7x+1}{e^{2x}}$$

$$\frac{\infty}{\infty} \checkmark$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{6x+7}{2e^{2x}} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{6}{4e^{2x}}$$

$$= 0$$

FACT

$$\lim_{x\to\infty} \frac{P_n(x)}{e^{ax}} = 0 \quad \text{where}$$

$P_n(x)$ = polynomial of degree n and $a > 0$.

Ex: $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ $\infty(0)$

Rewrite the function.

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \leftarrow \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= 1$$

ASIDE

Not indeterminate: $\frac{0}{\infty} = 0$

$$\frac{1}{0^+} = \infty \qquad \frac{1}{0^-} = -\infty$$

$$\infty + \infty = \infty$$

$$-\infty - \infty = -\infty$$

$\underline{\text{Ex}}$: $\displaystyle\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$

Form is $\infty - \infty$. Rewrite:

$$= \lim_{x\to 1^+} \frac{x-1-\ln x}{(\ln x)(x-1)} \leftarrow \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + (x-1)\left(\frac{1}{x}\right)} \leftarrow \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + (x-1)\left(\frac{-1}{x^2}\right) + \frac{1}{x}(1)}$$

$$= \frac{1}{2}$$

To deal with 0^0, ∞^0 and 1^∞ we use logarithms.

$\underline{\text{Ex}}$: $\displaystyle\lim_{x\to 0^+} x^x$

The form is 0^0.

$$\text{Let} \quad L = \lim_{x \to 0^+} x^x$$

$$\ln L = \ln\left(\lim_{x \to 0^+} x^x\right)$$

$$= \lim_{x \to 0^+} \ln x^x$$

$$= \lim_{x \to 0^+} x \ln x \qquad 0(-\infty)$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \qquad \frac{-\infty}{\infty}$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x \to 0^+} \frac{1}{x}\left(-x^2\right)$$

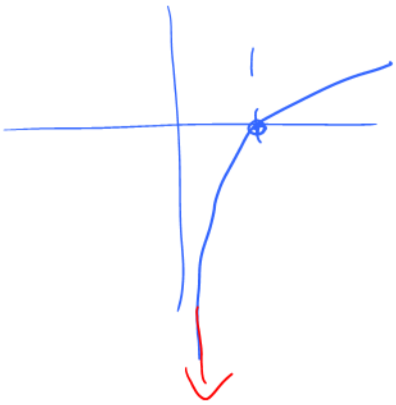
$$= \lim_{x \to 0^+} -x$$

$$= 0$$

$$\ln L = 0$$

$$L = e^0 = 1$$

$y = \ln x$



$$\lim_{x \to 0^+} \ln x = -\infty$$

Ex: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Form is 1^∞.

$$\text{Let } L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$\ln L = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$$

$\infty(0)$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

$\frac{0}{0}$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= 1$$

$$\ln L = 1$$

$$L = e$$

$\underline{Ex}$: $\lim_{x\to\frac{\pi}{2}^-} \tan x^{\cos x}$ $\leftarrow$ ∞^0

Let $L = \lim_{x\to\frac{\pi}{2}^-} \tan x^{\cos x}$

$$\ln L = \lim_{x\to\frac{\pi}{2}^-} \ln\left(\tan x^{\cos x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}^-} (\cos x)\ln(\tan x)$$

$\leftarrow 0(\infty)$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\tan x)}{\sec x} \leftarrow \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\tan x}(\sec^2 x)}{\sec x \tan x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cancel{\cos x}}\left(\frac{\cancel{\cos^2 x}}{\sin^2 x}\right) \quad \cos x$$

$$= \frac{0}{1}$$

$$= 0$$

$$\ln L = 0$$

$$L = e^0 = 1$$