8.5 Partial Fractions Cont'd

Ex: $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x+?)(x+?)$$
$$= x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{\cancel{2x^3+3x^2-2x}} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$x(2x-1)(x+2)$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$x=0$: $\quad -1 = A(-1)(2) \Rightarrow A = \frac{1}{2}$

$x=-2$: $\quad -1 = C(-2)(-5) \Rightarrow C = \frac{-1}{10}$

$x=\frac{1}{2}$: $\quad \frac{1}{4} = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$

$$\text{Integral} = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{(2x-1)} - \frac{1}{10}\frac{1}{(x+2)}\right]dx$$

Shortcut

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

If degree(numerator) $\geq$ degree(denominator)
then do long division.

Ex: $$\int \frac{2x^3-3x^2-3x-10}{x^2-2x-3}\,dx$$

$$
\begin{array}{r l}
 & 2x+1 \\
(x^2-2x-3) \,\big) & 2x^3 - 3x^2 - 3x - 10 \\
 & -(2x^3 - 4x^2 - 6x) \\ \hline
 & x^2 + 3x - 10 \\
 & -(x^2 - 2x - 3) \\ \hline
 & 5x - 7
\end{array}
$$

$$\text{Integrand} = 2x+1 + \frac{5x-7}{x^2-2x-3}$$

$$\frac{5x-7}{\cancel{x^2-2x-3}} = \frac{A}{x+1} + \frac{B}{x-3}$$
$(x+1)(x-3)$

$$5x-7 = A(x-3) + B(x+1)$$

$x=3$: $\quad 8 = B(4) \implies B=2$

$x=-1$: $\quad -12 = A(-4) \implies A=3$

Integrand $= 2x+1 + \frac{3}{x+1} + \frac{2}{x-3}$

Integral $= x^2 + x + 3\ln|x+1| + 2\ln|x-3| + C$

We've looked at denominators consisting of distinct linear factors.

Repeated Linear Factors

FACT

$$\frac{\text{polynomial}}{(x+1)(x+5)^2} = \frac{A}{x+1} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

$$\frac{\text{polynomial}}{(x+1)(x+5)^n} = \frac{A}{x+1} + \frac{A_1}{x+5} + \frac{A_2}{(x+5)^2} + \dots + \frac{A_n}{(x+5)^n}$$

(assuming long division is not required)

Ex: $\int \frac{2x}{(x+1)(x+2)^3}\,dx$

$$\frac{2x}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$2x = A(x+2)^3 + B(x+1)(x+2)^2 + C(x+1)(x+2) + D(x+1)$$

$x = -2$: $\quad -4 = D(-1) \Rightarrow D = 4$

$x = -1$: $\quad -2 = A$

x^3 Coefficient: $\quad 0 = A + B \Rightarrow B = 2$

Sub any number, $x = 0$:

$$0 = A(8) + B(4) + C(2) + D(1)$$

$$0 = -16 + 8 + 2C + 4$$

$$4 = 2C$$

$$C = 2$$

$$\text{Integral} = \int \left[\frac{-2}{x+1} + \frac{2}{x+2} + \frac{2}{(x+2)^2} + \frac{4}{(x+2)^3} \right] dx$$

$$= -2\ln|x+1| + 2\ln|x+2| - 2(x+2)^{-1} - 2(x+2)^{-2} + C$$

Drill

$$\int \frac{190}{u^2}\,du = -190u^{-1} + C$$

$$\int \frac{190}{(x+71)^2}\,dx = -190(x+71)^{-1} + C$$

$$\int \frac{190}{(2x+71)^2}\,dx = \frac{-190}{2}(2x+71)^{-1} + C$$

(could do substitution)

Irreducible Quadratic Factors

ax^2+bx+c is irreducible (can't be factored) if $b^2-4ac<0$

Ex: x^2+4 is irreducible

$x^2+4x+13$ "

FACT

$$\frac{\text{polynomial}}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

$$\frac{\text{polynomial}}{(x+3)(x^2+4)^2} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

(assuming long division is not required)

<u>Note</u>: Every polynomial can be factored into linear and irreducible quadratic factors.

<u>Ex</u>: $\displaystyle\int \frac{2x+1}{x^4+3x^2-4}\,dx$

$$x^4+3x^2-4 = (x^2+\,?\,)(x^2+\,?\,)$$
$$= (x^2-1)(x^2+4)$$
$$= (x-1)(x+1)(x^2+4)$$

$$\frac{2x+1}{(x-1)(x+1)(x^2+4)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+4}$$

$$2x+1 = A(x+1)(x^2+4) + B(x-1)(x^2+4) + (Cx+D)(x-1)(x+1)$$

$x=-1$: $\quad -1 = B(-2)(5) \Rightarrow B = \frac{1}{10}$

$x = 1$: $\quad 3 = A(2)(5) \Rightarrow A = \frac{3}{10}$

x^3 Coefficient: $\quad 0 = A + B + C$

$$0 = \frac{3}{10} + \frac{1}{10} + C$$

$$C = \frac{-4}{10} = \frac{-2}{5}$$

$x=0$: $\quad 1 = A(4) + B(-4) + (D)(-1)$

$$1 = \frac{12}{10} - \frac{4}{10} - D$$

$$D = \frac{12}{10} - \frac{4}{10} - \frac{10}{10}$$

$$D = \frac{-2}{10} = \frac{-1}{5}$$

$$\text{Integral} = \int \left[\frac{3}{10}\frac{1}{(x-1)} + \frac{1}{10}\frac{1}{(x+1)} - \frac{2}{5}\frac{x}{(x^2+4)} - \frac{1}{5}\frac{1}{(x^2+4)} \right] dx$$

$$= \frac{3}{10}\ln|x-1| + \frac{1}{10}\ln|x+1|$$

$$- \frac{1}{5}\ln|x^2+4| - \frac{1}{10}\tan^{-1}\frac{x}{2} + C$$