See Week 2 Monday tablet notes for which derivatives to know for Test 1.

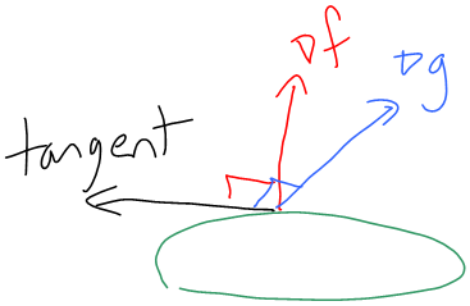
Test 1 covers 11.7, 12.1–12.7

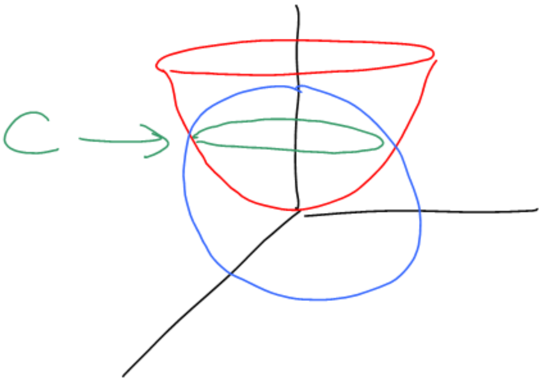
12.8 Cont'd

Ex: Let C be the intersection of $x^2+y^2=z$ and $\underbrace{x^2+y^2+z^2}_{g}=30$.

$\underbrace{x^2+y^2-z}_{f}=0$

Find the tangent vector to C at $(1,-2,5)$.





$$\text{tangent} = \nabla f \times \nabla g$$

$$\nabla f = [2x, 2y, -1]$$

$$\nabla f(1,-2,5) = [2, -4, -1]$$

$$\nabla g = [2x, 2y, 2z]$$

$$\nabla g(1,-2,5) = [2, -4, 10]$$

$$\text{tangent} = \nabla f \times \nabla g$$

$$= [-44, -22, 0]$$

or any nonzero multiple

$$\begin{array}{|ccccc} 2 & -4 & -1 & 2 & -4 \\ 2 & -4 & 10 & 2 & -4 \end{array}$$

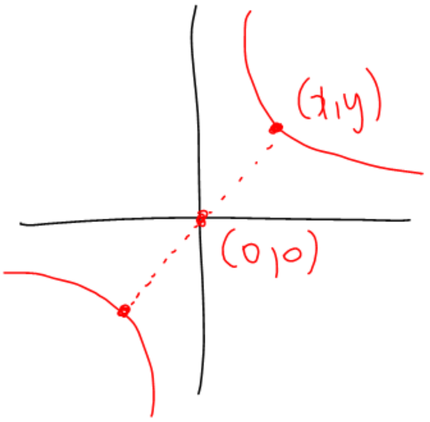
12.9 Lagrange Multipliers

Goal: Maximize or minimize a function f subject to a constraint $g = $ constant.

Method: Let $\nabla f = \lambda \nabla g$

λ is called the Lagrange multiplier.

Ex: Find the point(s) on $xy = 6$ that are closest to the origin.



$\underbrace{xy}_{g} = 6$

Let the point(s) be (x,y).

Minimize distance $d = \sqrt{x^2+y^2}$

Equivalent: Minimize $f = \underbrace{x^2+y^2}_{\text{distance}^2}$

This yields the same point(s) (x,y) as minimizing the distance.

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y] = \lambda [y, x]$$

① $2x = \lambda y \implies \lambda = \frac{2x}{y}$

② $2y = \lambda x \implies \lambda = \frac{2y}{x}$

③ $xy = 6$

$\lambda = \lambda$ (Solve for y in terms of x)

$$\frac{2x}{y} = \frac{2y}{x}$$

$$2x^2 = 2y^2$$

$$2y^2 = 2x^2$$

$$y^2 = x^2$$

$$\boxed{y = \pm x}$$

But ③ tells us that xy is positive.

$$\boxed{y = x}$$

$y = x \rightarrow$ ③ $xy = 6$

$$x^2 = 6$$

$$x = \pm\sqrt{6}$$

$x = \sqrt{6} \implies y = \sqrt{6}$

OR $x = -\sqrt{6} \implies y = -\sqrt{6}$

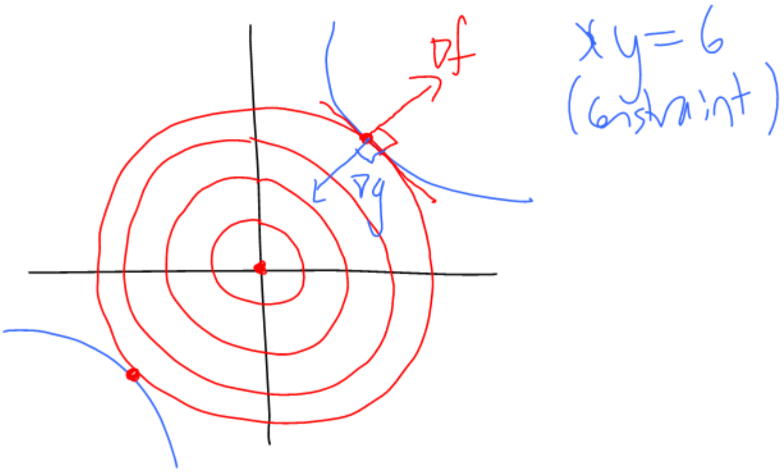
$$\boxed{\pm(\sqrt{6}, \sqrt{6})}$$

The Lagrange Multiplier method selects the level curve of f that is tangent to the constraint.

$f = x^2+y^2$

Level curves of f:

$x^2+y^2 = \text{constant}$



Curves are tangent

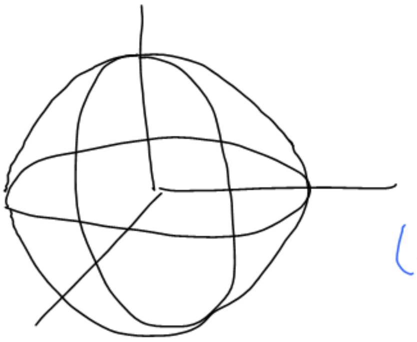
$\Rightarrow$ tangents are parallel

$\Rightarrow$ normals are parallel

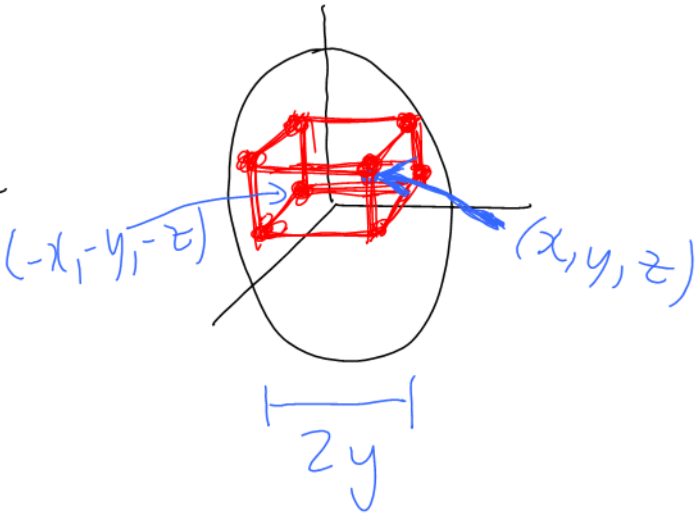
$\Rightarrow$ $\nabla f = \lambda \nabla g$

Ex: Find the maximum volume of a rectangular box that can be inscribed inside the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$



ellipsoid



$$\text{Maximize} \quad \text{Volume} = (2x)(2y)(2z)$$

$$= \underbrace{8xyz}_{f}$$

$$\text{Constraint:} \quad \underbrace{\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}}_{g} = 1$$

$$\nabla f = \lambda \nabla g$$

$$\vdots$$