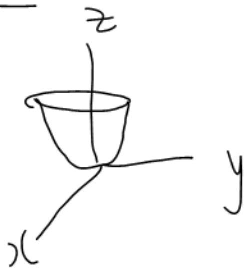
Assignment 1 Due
Mon Sept 22, 11:30 am

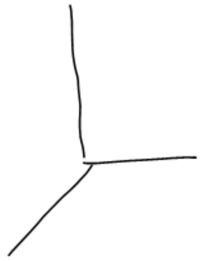
Explicit Functions

$z = x^2 + y^2$

or $f = x^2 + y^2$



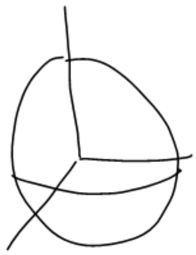
$$f = \frac{1}{1 + x^2 + y^2 + z^2}$$



e.g. f = temperature (°C)
at point (x, y, z)

Implicit Functions

$$\underbrace{x^2 + y^2 + z^2}_{f} = 4$$



12.8 Directional Derivatives Cont'd

Ex: Find the gradient of f

a) $f = x^2 y + 3xy^2$

$\nabla f = [f_x, f_y]$

$= [2xy + 3y^2, x^2 + 6xy]$

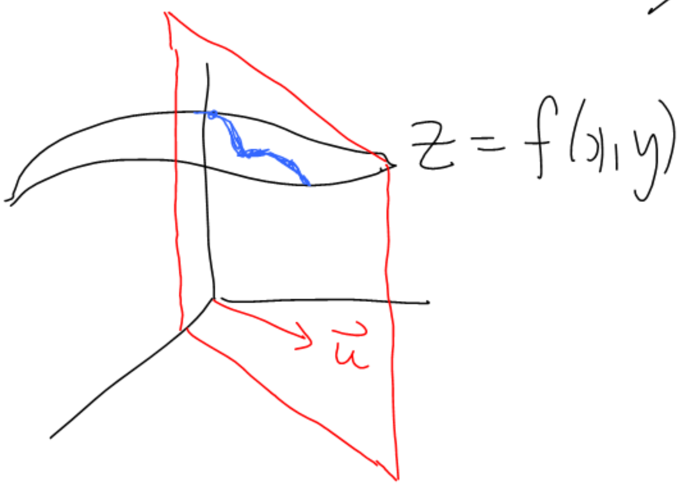
b) $f = xyz + x^2 z + y^3$

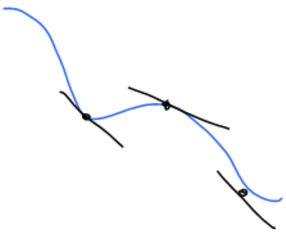
$\nabla f = [f_x, f_y, f_z]$

$= [yz + 2xz, xz + 3y^2, xy + x^2]$

Def

The directional derivative of f in direction $\vec{u}$ is written $D_{\vec{u}} f$.





$D_{\vec{u}} f$ = slope of the tangent line

FACT

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ where $\vec{u}$ is a unit vector.

Ex: Let $f = 8 - 9x - 6y - 2xy$ measure temperature (in °C).

Let x, y be position (in km).

From $(x,y) = (4,5)$ head towards $(2,-2)$.

a) Initial rate of change of f?

$$\text{direction} = [2-4, -2-5] = [-2, -7]$$

$$\vec{u} = \frac{1}{\sqrt{53}}[-2, -7]$$

$$\nabla f = [-9-2y, -6-2x]$$

$$\nabla f(4,5) = [-19, -14]$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{1}{\sqrt{53}}[-19, -14] \cdot [-2, -7] = \frac{136}{\sqrt{53}} \frac{°C}{km} \leftarrow \frac{\text{units of } f}{\text{units of } \sqrt{x^2+y^2}}$$

b) A train travels as above at $2 \frac{km}{min}$. Initial rate of change of f experienced by the train?

$$\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt}$$

rate of change of f w.r.t. distance

speed

$$= \frac{136}{\sqrt{53}} \frac{°C}{km}\left(2\ \frac{km}{min}\right)$$

$$= \frac{272}{\sqrt{53}}$$

$\frac{°C}{min}$ ← $\frac{\text{units of } f}{\text{units of time}}$

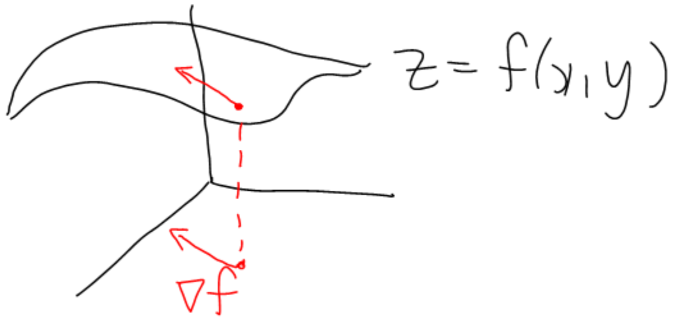
Quick Ex:

If $\vec{u} = [1,0]$ then $D_{\vec{u}}f = [f_x, f_y] \cdot [1,0]$
$= f_x$

If $\vec{u} = [0,1]$ then $D_{\vec{u}}f = [f_x, f_y] \cdot [0,1]$
$= f_y$

FACTS

1) ∇f points in the direction of (the steepest direction) the maximum rate of increase of f.

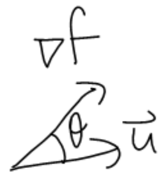


2) The maximum rate of increase of f is $\|\nabla f\|$.

Why?

1) $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \|\nabla f\| \, \|\vec{u}\| \cos\theta$



$D_{\vec{u}} f$ is maximized when $\vec{u}$ and ∇f are parallel.

Rephrased:

∇f points in the direction of the maximum $D_{\vec{u}} f$.

2) Suppose $\vec{u}$ is parallel to ∇f

$\Rightarrow \quad \vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$

$\Rightarrow \quad D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \nabla f \cdot \dfrac{\nabla f}{\|\nabla f\|}$

$\boxed{\vec{w} \cdot \vec{w} = \|\vec{w}\|^2}$

$= \dfrac{\|\nabla f\|^2}{\|\nabla f\|}$

$$= \|\nabla f\|$$

Ex: $f = x^3 + y^2 + 4z \quad (°C)$

Point $P = (1, 2, 3)$.

Let x, y, z be position (m).

a) From P, in which direction does f increase fastest?

$$\nabla f = [3x^2, 2y, 4]$$

$$\nabla f(P) = [3, 4, 4]$$

b) From P, what is the maximum rate of increase of f?

$$\|\nabla f(P)\|$$

$$= \|[3,4,4]\|$$

$$= \sqrt{41} \ \frac{°C}{m}$$

<u>Level curve</u> $\quad f(x,y) = \text{constant}$

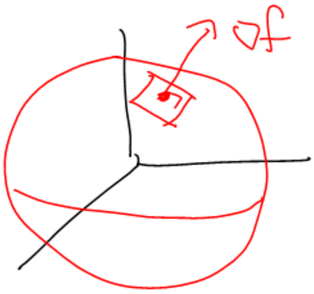
<u>Level surface</u> $\quad f(x,y,z) = \text{constant}$

FACT

If $f(x,y) = \text{constant}$ then $\nabla f \perp$ curve

FACT

If $f(x,y,z) = \text{constant}$ then $\nabla f \perp \text{surface}$



<u>Ex</u>: Given $x^2 + 4y^2 + 2z^2 = 25$.
Find the tangent plane to the surface at $(1,-2,2)$.

$\vec{n} = [-z_x, -z_y, 1]$ would be inconvenient.

Let $f = x^2 + 4y^2 + 2z^2$

Level surface $\Rightarrow$ $\nabla f \perp$ surface

$$\vec{n} = \nabla f = [2x, 8y, 4z] = [2, -16, 8]$$

Normal form $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$$\begin{bmatrix} 2 \\ -16 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -16 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$