12.7 Multivariable Chain Rule

Single Variable Chain Rule:

Suppose y depends on x,
and suppose x depends on t.

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

$$\begin{tikzcd} y \arrow[d, dash] \\ x \arrow[d, dash] \\ t \end{tikzcd}$$

The Multivariable Chain Rule
is a <u>process</u> rather than a <u>formula</u>.

<u>Ex</u>:

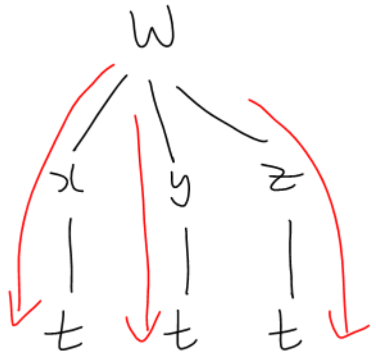
$$\begin{tikzcd}[column sep=small] & & & w \arrow[dll, dash] \arrow[drr, dash] & & & \\ & x \arrow[dl, dash] \arrow[dr, dash] & & & & y \arrow[dl, dash] \arrow[dr, dash] & \\ u & & v & & u & & v \end{tikzcd}$$

Find $\frac{\partial w}{\partial v}$.

$$\begin{tikzcd}[column sep=small] & & & w \arrow[dll, dash] \arrow[drr, dash] & & & \\ & x \arrow[dl, dash] \arrow[dr, dash] & & & & y \arrow[dl, dash] \arrow[dr, dash] & \\ u & & v & & u & & v \end{tikzcd}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial v}$$

Ex:



Find $\frac{dw}{dt}$.

e.g.

$w = xy + z^2$

$x = 1 + 2t$

$y = \sqrt{t}$

$z = 3t$

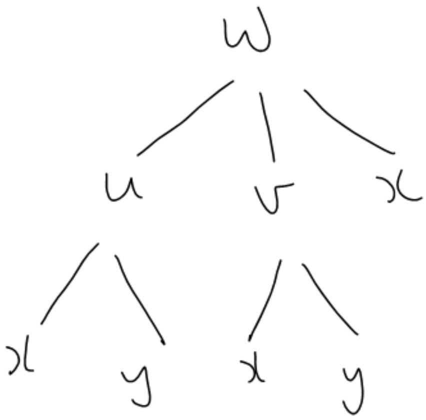
$\Rightarrow w = (1+2t)\sqrt{t} + (3t)^2$

partial derivative

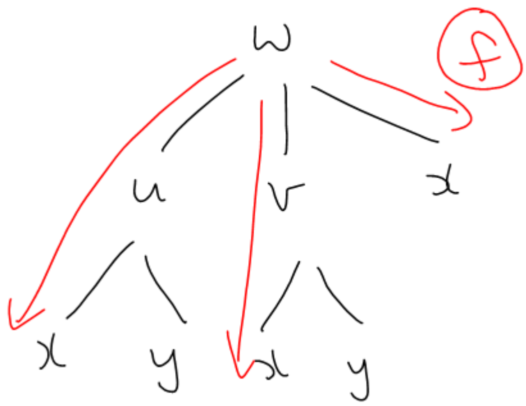
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

derivative

Ex:



Find $\frac{\partial w}{\partial x}$.



Caution: x appears on multiple levels.

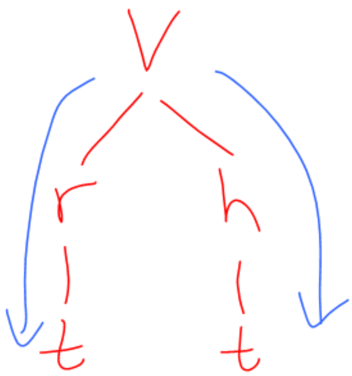
Let $w = f(u, v, x)$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}$$

CAUTION
This would be incorrect:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}$$

Ex: A cylinder's volume is decreasing such that $\frac{dh}{dt} = -3\ \frac{cm}{s}$ and $\frac{dr}{dt} = -1\ \frac{cm}{s}$. Find $\frac{dV}{dt}$ when $r = 3\,cm$ and $h = 8\,cm$.



$$\boxed{V = \pi r^2 h}$$

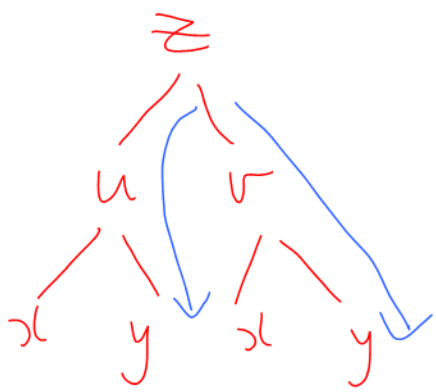
$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{dV}{dt} = (2\pi r h)\frac{dr}{dt} + (\pi r^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 48\pi(-1) + 9\pi(-3)$$

$$= -75\pi \ \frac{cm^3}{s}$$

<u>Ex</u>: Let $z = f(u,v)$,
$u = 3x + y^3$ and $v = xy - 3y^2$.
Given $z_u = -3$ and $z_v = 4$ at $(u,v) = (11,-10)$.
Find z_y at $(x,y) = (1,2)$.



$$z_y = z_u u_y + z_v v_y$$

$$z_y = z_u(3y^2) + z_v(x - 6y)$$

$$\boxed{\begin{aligned} (x,y) = (1,2) &\Rightarrow (u,v) = (11,-10) \\ &\Rightarrow z_u = -3,\ z_v = 4 \end{aligned}}$$

$$z_y = -3(12) + 4(-11)$$

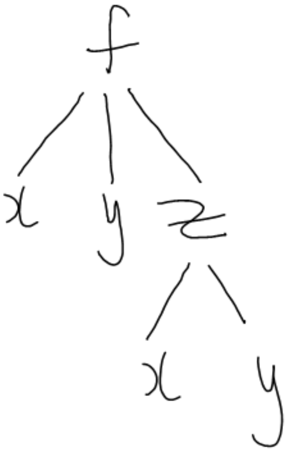
$z_y = -80$

Multivariable implicit function

e.g. $\underbrace{x^2+y^2+z^2+6xyz^3-2x+y}_{f}=0$

Context

- z depends on x and y
- x and y are independent



Warm Up:

$$\frac{\partial}{\partial x}[x^2] = 2x \qquad \frac{\partial}{\partial y}[x^2] = 0$$

$$\frac{\partial}{\partial x}[y^2] = 0 \qquad \frac{\partial}{\partial y}[y^2] = 2y$$

$$\frac{\partial}{\partial x}[z^2] = 2z\frac{\partial z}{\partial x} \qquad \frac{\partial}{\partial y}[z^2] = 2z\frac{\partial z}{\partial y}$$

Chain Rule   Chain Rule

Ex: Find $\frac{\partial z}{\partial y}$ given:

$y(6xz^3)$

$$x^2+y^2+z^2+6xyz^3-2x+y=0$$

Take $\frac{\partial}{\partial y}$:

$$0+2y+2z\frac{\partial z}{\partial y}+\underbrace{y\left(18xz^2\frac{\partial z}{\partial y}\right)+1(6xz^3)}_{\text{Product Rule}}+1=0$$

$$2z\frac{\partial z}{\partial y}+18xyz^2\frac{\partial z}{\partial y}=-2y-6xz^3-1$$

$$\left[2z+18xyz^2\right]\frac{\partial z}{\partial y}=-(2y+6xz^3+1)$$

$$\frac{\partial z}{\partial y}=\frac{-(2y+6xz^3+1)}{2z+18xyz^2}$$

This process is called "multivariable implicit differentiation."

Ex: Find $\frac{\partial z}{\partial x}$: $x(6yz^3)$

$$x^2+y^2+z^2+6xyz^3-2x+y=0$$

Take $\frac{\partial}{\partial x}$:

$$2x+2z\frac{\partial z}{\partial x}+\underbrace{x\left(18yz^2\frac{\partial z}{\partial x}\right)+1(6yz^3)}_{\text{Product Rule}}-2=0$$

$$\left[2z+18xyz^2\right]\frac{\partial z}{\partial x}=2-6yz^3-2x$$

$$\frac{\partial z}{\partial x}=\frac{2-6yz^3-2x}{2z+18xyz^2}$$

12.8 Directional Derivatives

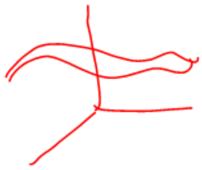
Def

The gradient of a function f :

$$\nabla f(x,y)=[f_x, f_y]$$

$$\nabla f(x,y,z)=[f_x, f_y, f_z]$$

The symbol ∇ is pronounced "del"



<u>Ex</u>: Find ∇f:

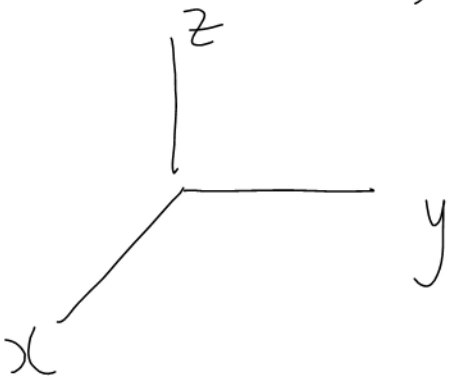
a) $f = x^2 + xy^3$

$\nabla f = [2x + y^3 , \; 3xy^2]$

b) $f = xy + yz^4$

$\nabla f = [\; y \;, \; x + z^4 \;, \; 4yz^3 \;]$

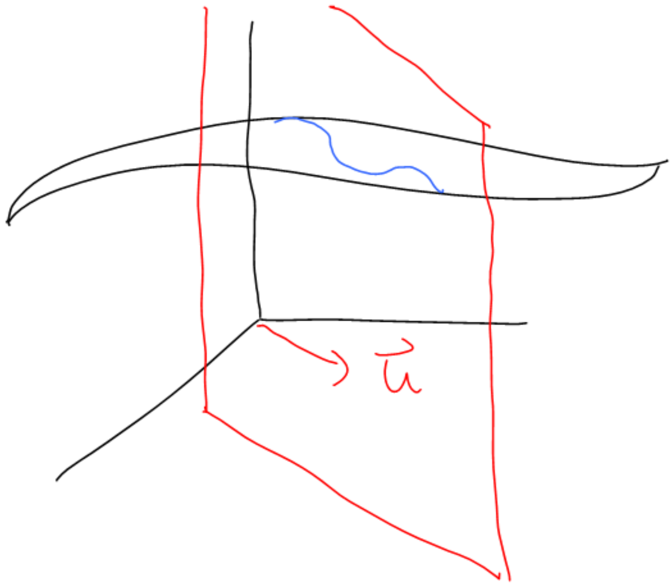
Context for b)



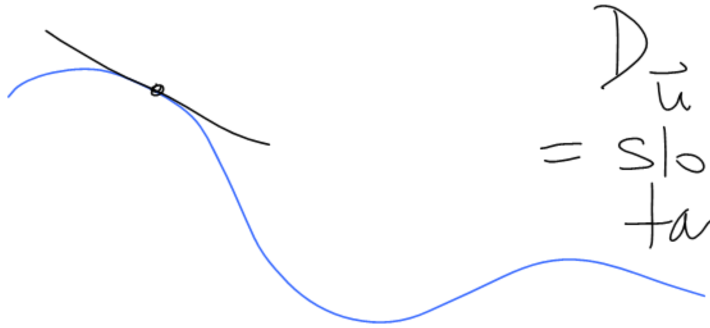
f = temperature at the point (x, y, z)

Def

The <u>directional derivative of f</u> in direction $\vec{u}$ is $D_{\vec{u}} f$.



surface
$z = f(x,y)$



$D_{\vec{u}} f$
= slope of the tangent line