8.1 Basic Trig Integrals Cont'd

Ex: Find $\int \frac{1}{\sqrt{x}\sin\sqrt{x}}\,dx$

$$= \int \frac{\csc\sqrt{x}}{\sqrt{x}}\,dx$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \tfrac{1}{2}x^{-1/2}\,dx \\ 2\,du &= \frac{dx}{\sqrt{x}} \end{aligned}}$$

$$= 2\int \csc u\,du$$

$$= -2\ln|\csc u + \cot u| + C$$

$$= -2\ln|\csc\sqrt{x} + \cot\sqrt{x}| + C$$

Ex: $\int \frac{e^x \cos e^x}{\sin e^x}\,dx$

$$= \int e^x \cot e^x \, dx$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x \, dx \end{aligned}}$$

$$= \int \cot u \, du$$

$$= -\ln|\csc u| + C$$

$$= -\ln|\csc e^x| + C$$

Ex: $\displaystyle \int \frac{\sec(\ln x)\tan(\ln x)\, dx}{x}$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}}$$

$$= \int \sec u \tan u \, du$$

$$= \sec u + C$$

$$= \sec(\ln x) + C$$

Recall: lo trig integrals

POWER FORM $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

LN FORM $\int \frac{1}{x}\,dx = \ln|x| + C$

EXP FORM $\int e^x\,dx = e^x + C$

ARCTAN " $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

ARCSINE " $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a} + C$

Mixed Practice

Ex: Find $\int \dfrac{(1+\ln x)^3}{x}\,dx$

$$\boxed{\begin{aligned} u &= 1+\ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{1}{4}(1+\ln x)^4 + C$$

Ex: Find

a) $\displaystyle\int \frac{x}{9+x^2}\,dx$

$$\boxed{\begin{aligned} u &= 9+x^2 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|9+x^2| + C$$

b) $\displaystyle\int \frac{x}{9+x^4}\,dx$

$$= \int \frac{x}{9+(x^2)^2}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{3^2+u^2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\frac{u}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\frac{x^2}{3} + C$$

$\underline{Ex}$: $\displaystyle\int (\sec^2 x)\, e^{\tan x}\, dx$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\, dx \end{aligned}}$$

$$= \int e^u\, du$$

$$= e^u + C$$

$$= e^{\tan x} + C$$

$$\int e^x\, dx = e^x + C \qquad \int e^{2x}\, dx = \frac{e^{2x}}{2} + C$$

Ex: Find $\displaystyle\int \frac{e^{4x}}{\sqrt{9-e^{8x}}}\,dx$

$$= \int \frac{e^{4x}\,dx}{\sqrt{3^2-(e^{4x})^2}}$$

$$\boxed{\begin{aligned} u &= e^{4x} \\ du &= 4e^{4x}\,dx \\ \frac{du}{4} &= e^{4x}\,dx \end{aligned}}$$

$$= \frac{1}{4}\int \frac{du}{\sqrt{3^2-u^2}}$$

$$= \frac{1}{4}\sin^{-1}\frac{u}{3} + C$$

$$= \frac{1}{4}\sin^{-1}\frac{e^{4x}}{3} + C$$

__Ex:__ Six similar integrals

a) $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$

b) $\displaystyle\int \frac{x}{1+x^2}\,dx$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$\displaystyle = \frac{1}{2}\int \frac{du}{u}$

$\displaystyle = \frac{1}{2}\ln|u| + C$

$\displaystyle = \frac{1}{2}\ln|1+x^2| + C$

c) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

$$\boxed{u = 1+x^2}$$

$\displaystyle = \frac{1}{2}\int u^{-1/2}\,du$

$\displaystyle = \frac{1}{2}\left(2u^{1/2}\right) + C$

$= \sqrt{1+x^2} + C$

d) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

e) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \\ -\tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\left(2u^{1/2}\right) + C$

$= -\sqrt{1-x^2} + C$

f) $\int \frac{x}{1-x^2}\,dx$

$$\boxed{u = 1-x^2}$$

$= -\frac{1}{2}\int \frac{du}{u}$

$= -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln|1-x^2| + C$

Ex: Integrals that come up often

$\int e^x\,dx = e^x + C$

$\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \quad (k \neq 0)$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos 4x\,dx = \frac{1}{4}\sin 4x + C$$

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C$$

$$(k \neq 0)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sin kx\,dx = \frac{-1}{k}\cos kx + C$$

$$(k \neq 0)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$(a \neq 0)$$

$$\int \frac{dx}{2x+3} = \frac{1}{2}\ln|2x+3| + C$$

ASIDE $\int \frac{x\,dx}{2x+3}$ Long Division

8.2 Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

Why? $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$

Apply $\int dx$: $uv = \int u\frac{dv}{\cancel{dx}}\cancel{dx} + \int v\frac{du}{\cancel{dx}}\cancel{dx}$

$$uv - \int v\,du = \int u\,dv$$

$$\int u\,dv = uv - \int v\,du$$

Ex: $\int x e^{2x} dx$

$$\boxed{\begin{array}{ll} u = x & dv = e^{2x} dx \\ du = dx & v = \frac{e^{2x}}{2} \end{array}}$$

Don't write $+C$

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^{2x} dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

Ex: $\int x^3 \ln x \, dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^3 dx \\ du = \frac{1}{x} dx & v = \frac{x^4}{4} \end{array}}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$