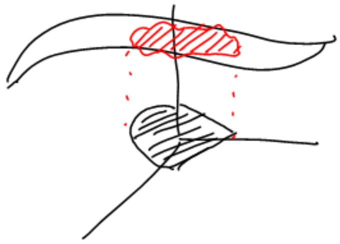
12.5 Cont'd

Ex Cont'd



1) Interior Critical Points

$\left(-\frac{1}{2}, 0\right)$

2) Corners

$(-1,0), (0,1), (1,0)$

3) Critical Points on Side 1: $y = x+1$

$\left(-\frac{3}{4}, \frac{1}{4}\right)$

4) Critical Points on Side 2: $y = 1-x$

$y = 1-x \rightarrow \quad f = x + x^2 + y^2$

$f = x + x^2 + (1-x)^2$

$f' = 1 + 2x + \cancel{2}_{-2}(1-x)\cancel{(-1)}$

$f' = 1 + 2x - 2 + 2x$

$f' = 4x - 1$

Set $f' = 0$: $\quad 4x - 1 = 0$

$x = \frac{1}{4}$

$y = 1 - x = \frac{3}{4}$

$\left(\frac{1}{4}, \frac{3}{4}\right)$

5) Critical Points on Side 3: $x^2+y^2=1$

$$f = x + \underbrace{x^2+y^2}_{1}$$

$$f = x+1$$

$$f' = 1$$

$$f' \neq 0$$

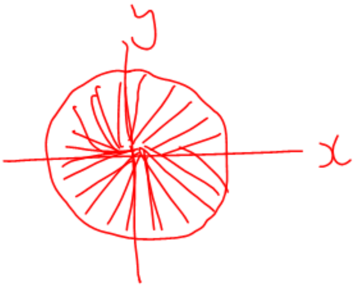
No critical points on Side 3.

Points	$f = x+x^2+y^2$	
$\left(-\frac{1}{2}, 0\right)$	$-\frac{1}{4}$	$\Leftarrow$ absolute min
$(-1, 0)$	0	
$(0, 1)$	1	
$(1, 0)$	2	$\Leftarrow$ absolute max
$\left(-\frac{3}{4}, \frac{1}{4}\right)$	$-\frac{1}{8}$	
$\left(\frac{1}{4}, \frac{3}{4}\right)$	$\frac{7}{8}$	

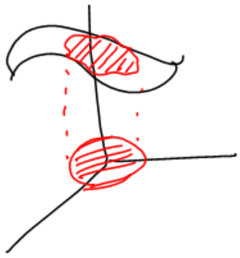
The absolute minimum of f is $-\frac{1}{4}$, achieved at $(x,y) = \left(-\frac{1}{2}, 0\right)$.

The absolute maximum of f is 2, achieved at $(x,y) = (1,0)$.

Ex: Find the absolute maximum of $z = x^2y^2$ over the region $x^2+y^2 \leq \frac{1}{3}$.



$x^2+y^2 \le \frac{1}{3}$



1) Interior Critical Points

$z_x = 2xy^2$

$z_y = 2x^2y$

} both 0 or undefined

$2xy^2 = 0$ and $2x^2y = 0$

$x=0$ or $y=0$ $\quad$ $x=0$ or $y=0$

$\{(x,y)$ such that $x^2+y^2<\frac{1}{3}$, and $x=0$ or $y=0\}$

$\boxed{\text{A circle has 0 corners and 1 side.}}$

2) Critical Points on Side 1: $x^2+y^2=\frac{1}{3}$

$y^2=\frac{1}{3}-x^2 \longrightarrow z=x^2y^2$

$z=x^2\left(\frac{1}{3}-x^2\right)$

$z=\frac{x^2}{3}-x^4$

$z'=\frac{2x}{3}-4x^3$

Set $z' = 0$: $\frac{2x}{3} - 4x^3 = 0$

$2x\left(\frac{1}{3} - 2x^2\right) = 0$

$x = 0$ (already discussed)

$\frac{1}{3} - 2x^2 = 0$

$\frac{1}{3} = 2x^2$

$x^2 = \frac{1}{6}$

$x = \pm\frac{1}{\sqrt{6}}$

$y^2 = \frac{1}{3} - x^2$

$y^2 = \frac{1}{3} - \frac{1}{6}$

$y^2 = \frac{1}{6}$

$y = \pm\frac{1}{\sqrt{6}}$

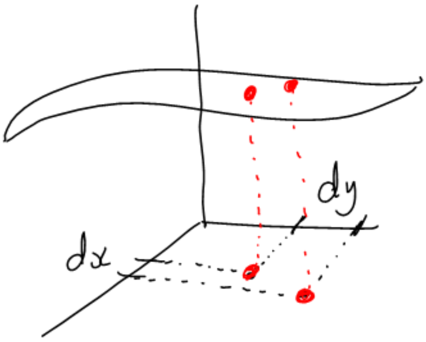
$\left(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}\right)$

Points	$z = x^2y^2$
$x=0$ or $y=0$	0
$\left(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}\right)$	$\frac{1}{36}$ ← absolute maximum

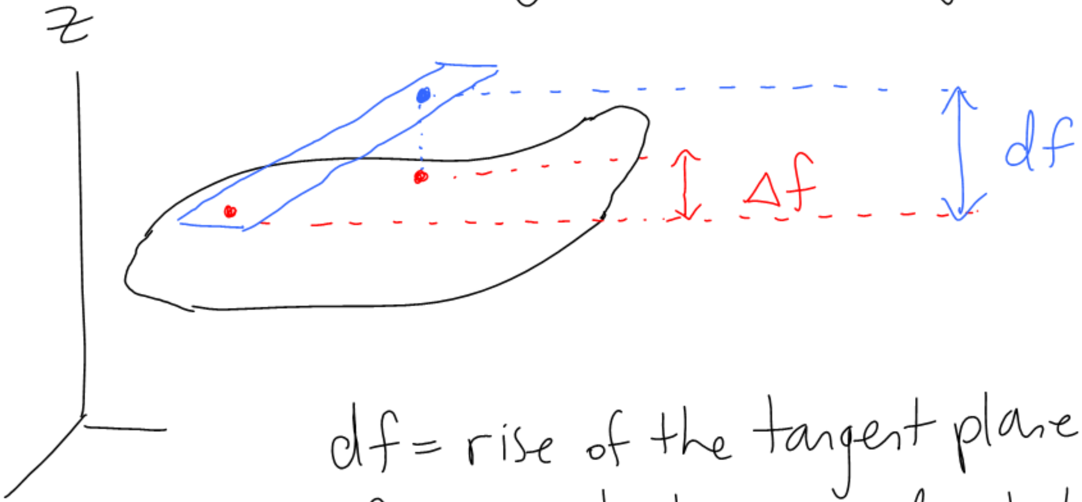
The absolute maximum of z is $\frac{1}{36}$, achieved at $(x,y) = \left(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}\right)$.

12.6 Differentials and Linear Approximation

The differential of a function f is $df = f_x\,dx + f_y\,dy$



dx and dy: small changes in x and y



df = rise of the tangent plane between 2 points

Δf = actual change in f between 2 points

FACT $\Delta f \approx df$ when dx and dy are small.

Ex: The range of a projectile

$$R = \frac{V_0^2 \sin 2\theta}{9.8}$$

Approximate the change in R if
θ increases from 30.0° to 30.3°
and V_0 " from 2.00 m/s to 2.10 m/s.

$$df = f_x\,dx + f_y\,dy$$

$$dR = R_{v_0}\,dv_0 + R_\theta\,d\theta$$

$$dR = \frac{2v_0 \sin 2\theta}{9.8}\,dv_0 + \frac{2v_0^2 \cos 2\theta}{9.8}\,d\theta$$

Sub $v_0 = 2.00 \qquad \theta = 30.0^\circ$

$dv_0 = 0.10 \qquad d\theta = 0.3^\circ = \frac{0.3\pi}{180}$

$$dR = \frac{4.00\left(\frac{\sqrt{3}}{2}\right)}{9.8}(0.10) + \frac{8.00\left(\frac{1}{2}\right)}{9.8}\left(\frac{0.3\pi}{180}\right)$$

$$\approx 0.037 \text{ m} \checkmark$$

$$\Delta R \approx dR$$

$$\approx 0.037 \text{ m} \checkmark$$

Ex: The point $(2,1)$ lies on the curve $3x^2+3y^2-4xy-7=0$. Approximate the x-coordinate of the point on the curve with $y=0.9$

$(2+dx, 0.9)$ $(2,1)$

Let $f = 3x^2 + 3y^2 - 4xy - 7$

A point has $f=0 \iff$ the point is on the curve

$$df = f_x\,dx + f_y\,dy$$

$$df = (6x - 4y)\,dx + (6y - 4x)\,dy$$

Sub

$\Delta f = 0$ (both points have $f=0$)

$df \approx \Delta f$

$df \approx 0$

$x = 2$ $\quad$ $y = 1$

$dx = ?$ $\quad$ $dy = -0.1$

messy-nice

$$0 \approx 8\,dx - 2(-0.1)$$

$$-0.2 \approx 8\,dx$$

$$dx \approx -0.025$$

$$2 + dx \approx 1.975$$