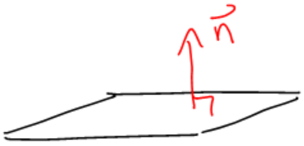
Assignment 1 is on website.
Due Mon Sept 22, 11:30am
Submit on D2L

12.4 Cont'd

Recall $\vec{n} = [-z_x, -z_y, 1]$

Question 1: Where is a tangent plane horizontal?

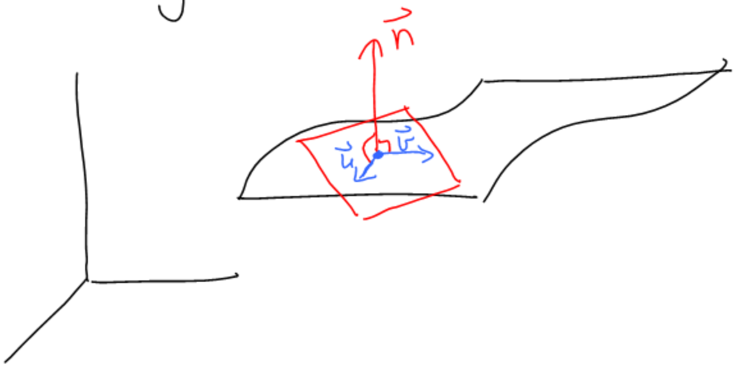


Horizontal tangent plane

$\iff \vec{n} = [0, 0, 1]$

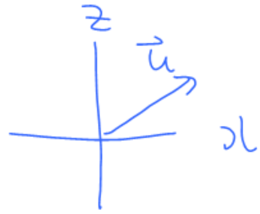
$\iff z_x = 0$ and $z_y = 0$

Question 2: Why does $\vec{n} = [-z_x, -z_y, 1]$?

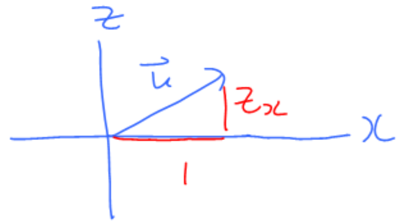


Let $\vec{u}$ and $\vec{v}$ be direction vectors for the tangent plane parallel to the x-axis and the y-axis respectively.

Slope of $\vec{u}$ = rate of change of z w.r.t. x

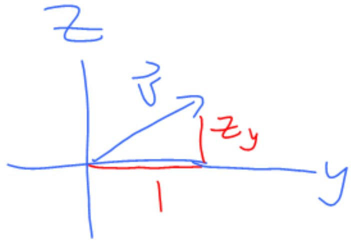


$= z_x$



$\vec{u} = [1, 0, z_x]$

Slope of $\vec{v}$ = rate of change of z w.r.t. y



$= z_y$

$\vec{v} = [0, 1, z_y]$

$\vec{n} = \vec{u} \times \vec{v}$

$$\boxed{\begin{array}{ccccc} 1 & 0 & z_x & 1 & 0 \\ 0 & 1 & z_y & 0 & 1 \end{array}}$$

$= [-z_x, -z_y, 1]$

Ex: Find all points (x, y, z) where the tangent plane is horizontal:

$$z = x^3 - 12x + y^2 + 8y + 10.$$

$z_x = 3x^2 - 12$ $\qquad$ $z_y = 2y + 8$

$z_x = 0$:

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

AND

$z_y = 0$:

$2y + 8 = 0$

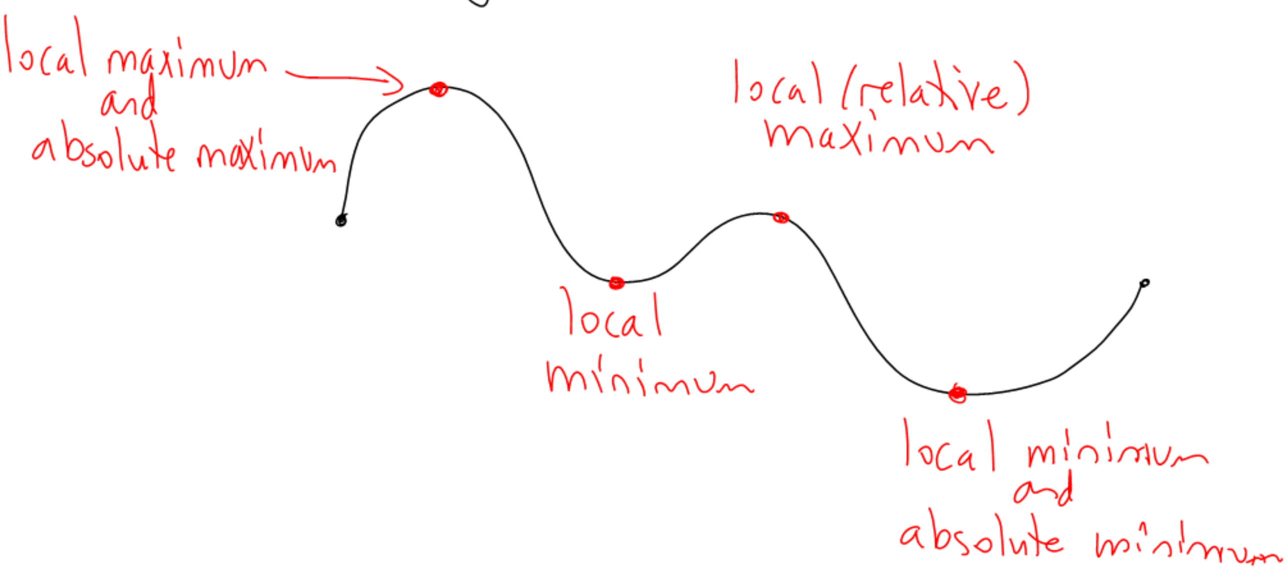
$y = -4$

$(x, y, z) = (2, -4, -22), \quad (-2, -4, 10)$

given z

12.5 Multivariable Optimization

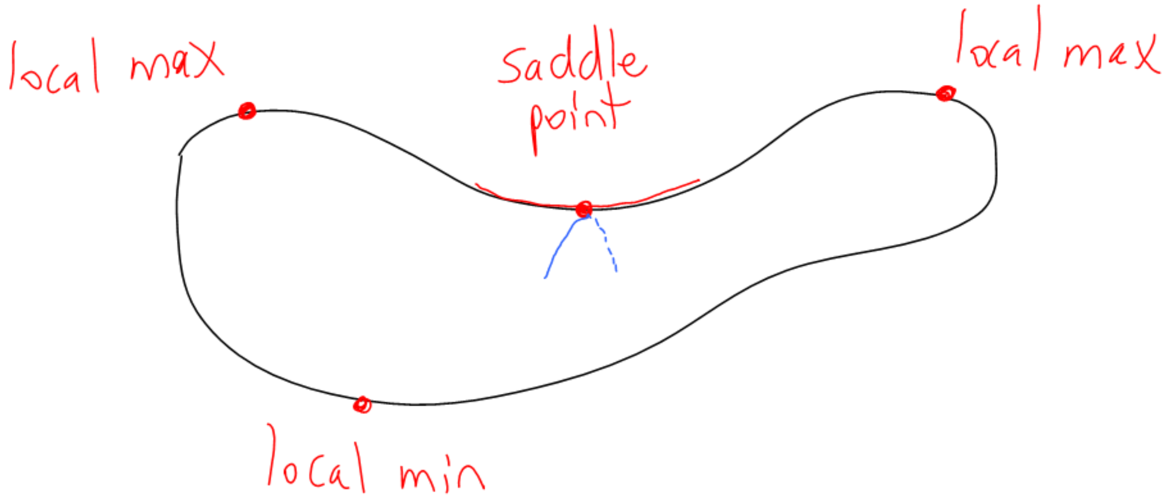
From single-variable calculus:



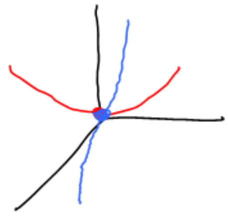
Def

Points where z_x and z_y are both 0 or undefined are called <u>critical points</u>.

- Could have $z_x = 0$ and z_y undefined, or vice versa.
- Critical points can be a local maximum, local minimum, saddle point, or none of the above.



Weirder critical point:



Ex: Find all critical points of
$$f = x^3 + xy^2 + 3x^2 - 3y^2$$

$$\left.\begin{aligned} f_x &= 3x^2 + y^2 + 6x \\ f_y &= 2xy - 6y \end{aligned}\right\} \text{both } 0 \text{ or undefined}$$

$f_x = 0$: $\quad 3x^2 + y^2 + 6x = 0 \quad$ (1)

$f_y = 0$: $\quad 2xy - 6y = 0 \quad$ (2)

② : $2xy - 6y = 0$

$2y(x-3) = 0$

$y = 0$ or $x = 3$ (2 cases)

Case 1: $y = 0$

$y = 0 \rightarrow$ ① : $3x^2 + y^2 + 6x = 0$

$3x^2 + 6x = 0$

$3x(x+2) = 0$

$x = 0, -2 \implies (0,0), (-2,0)$

Case 2: $x = 3$

$x = 3 \rightarrow$ ① : $3x^2 + y^2 + 6x = 0$

$27 + y^2 + 18 = 0$

$y^2 = -45$

no real solutions

$\boxed{(0,0), (-2,0)}$ ✓

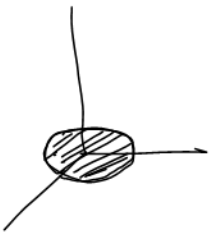
or $(0,0,0), (-2,0,4)$ ✓

$\uparrow f \qquad \uparrow f$

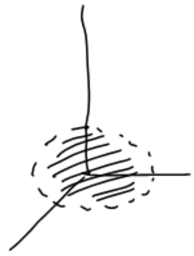
FACT

A continuous function defined over a closed domain attains an absolute maximum and an absolute minimum.

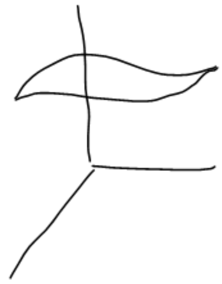
region of the xy-plane that includes its boundary



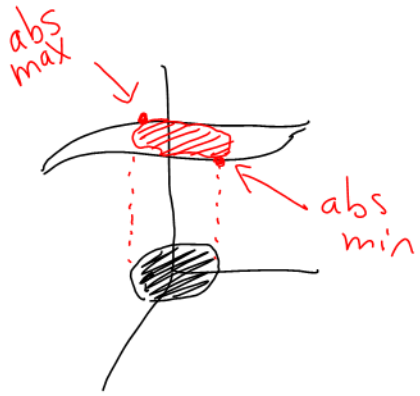
closed domain



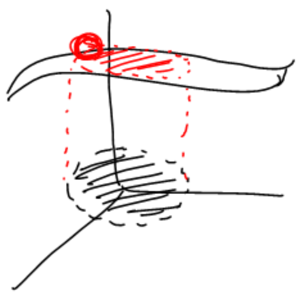
non-closed domain



surface



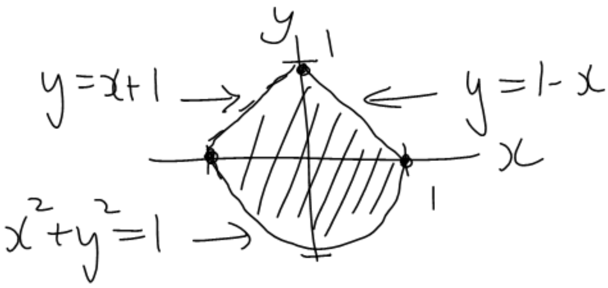
Can fail if the domain is not closed:

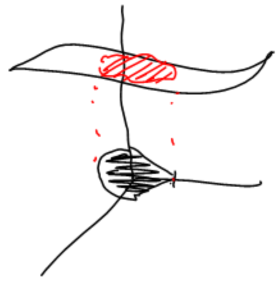


FACT

The absolute maximum and absolute minimum of f may occur in the interior of the domain, at a corner of the domain, or on a side of the domain.

Ex: Find the absolute maximum and absolute minimum of $f = x + x^2 + y^2$ over the region below:





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1) Interior Critical Points

$f_x = 1 + 2x$

$f_y = 2y$

both 0 or undefined

$1 + 2x = 0 \Rightarrow x = -\frac{1}{2}$

$2y = 0 \Rightarrow y = 0$

$\left(-\frac{1}{2}, 0\right)$ This is in the domain. ✓

2) Corners

$(-1, 0), (1, 0), (0, 1)$

3) Critical Points on Side 1: $y = x + 1$

$y = x + 1 \rightarrow f = x + x^2 + y^2$

$f = x + x^2 + (x+1)^2$ single-variable function

$$f' = 1 + 2x + 2(x+1)$$
$$= 4x + 3$$

Set $f' = 0$: $\quad 4x + 3 = 0$

$$x = -\frac{3}{4}$$

$$y = x + 1 = \frac{1}{4}$$

$$\left(-\frac{3}{4}, \frac{1}{4}\right)$$

4) Critical Points on Side 2: $y = 1 - x$

⋮

5) Critical Points on Side 3: $x^2 + y^2 = 1$

⋮

Points	$f = x + x^2 + y^2$

TO BE CONTINUED