12.5 Arc Length Cont'd

Ex: Find the arc length of
$\bar{r}(t) = [2t^2+1,\ t^3+5]$ on $0 \le t \le 1$.

$\vdots$

$$s = \int_0^1 t\sqrt{16+9t^2}\,dt$$

$u = 16+9t^2$
$du = 18t\,dt$
$\frac{1}{18}du = t\,dt$
$t=0 \Rightarrow u=16$
$t=1 \Rightarrow u=25$

$$= \frac{1}{18}\int_{16}^{25} \sqrt{u}\,du$$

$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{16}^{25}$$

$$= \frac{1}{27}\left[u^{3/2}\right]_{16}^{25}$$

$$= \frac{1}{27}(125-64)$$

$$= \frac{61}{27}$$

Test Review

Ex: Find the first 3 nonzero terms of the Maclaurin series for:

a) $\dfrac{1}{\sqrt{1-x^2}}$

$$(1+x)^k \approx 1 + kx + \frac{k(k-1)}{2!}x^2$$

$$(1+x)^{-1/2} \approx 1 - \tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)x^2$$

$$\approx 1 - \frac{x}{2} + \frac{3x^2}{8}$$

$$(1-x^2)^{-1/2} \approx 1 - \frac{(-x^2)}{2} + \frac{3(-x^2)^2}{8}$$

$$\approx 1 + \frac{x^2}{2} + \frac{3x^4}{8}$$

b) $(1+x)e^{2x}$

$$e^x \approx 1 + x + \frac{x^2}{2}$$

$$e^{2x} \approx 1 + 2x + \cancel{\frac{(2x)^2}{2}}\ 2x^2$$

$$xe^{2x} \approx x + 2x^2 + \ldots$$

$$(1+x)e^{2x} = e^{2x} + xe^{2x}$$

$$\approx 1 + 2x + 2x^2$$
$$\quad\quad\quad x + 2x^2$$

$$\approx 1 + 3x + 4x^2$$

Ex: Find (x,y) where the tangent is horizontal or vertical:

$$\begin{cases} x = t^2 - 4t + 2 \\ y = t^3 - 3t \end{cases}$$

$$\frac{dx}{dt} = 2t - 4$$

Set $\frac{dx}{dt} = 0$: $0 = 2t - 4$

$t = 2$

$$\frac{dy}{dt} = 3t^2 - 3$$

Set $\frac{dy}{dt} = 0$: $0 = 3t^2 - 3$

$0 = 3(t^2 - 1)$

$0 = 3(t-1)(t+1)$

$t = \pm 1$

Horizontal Tangent $\Rightarrow$ $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\Rightarrow$ $t = \pm 1$

$\Rightarrow$ $(x, y) = (-1, -2)$ or $(7, 2)$

Vertical Tangent $\Rightarrow$ $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\Rightarrow$ $t = 2$

$\Rightarrow$ $(x, y) = (-2, 2)$

Ex: Find the slope of the tangent line to the curve below at $\theta = \frac{\pi}{3}$:

$$r = 1 + \sin\theta$$

$x = r\cos\theta$

$x = (1 + \sin\theta)\cos\theta$

$$\frac{dx}{d\theta} = (1+\sin\theta)(-\sin\theta) + \cos^2\theta$$

$$y = r\sin\theta$$

$$y = (1+\sin\theta)\sin\theta$$

$$y = \sin\theta + \sin^2\theta$$

$$\frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{\cos\theta + 2\sin\theta\cos\theta}{(1+\sin\theta)(-\sin\theta) + \cos^2\theta}$$

$$\boxed{\begin{aligned} \cos\frac{\pi}{3} &= \frac{1}{2} \\ \sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2} \end{aligned}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\frac{1}{2} + 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(1+\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{4}}$$

$$= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \frac{3}{4} + \frac{1}{4}}$$

$$= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \frac{1}{2}}$$

$$= -1$$

Ex: Find $\dfrac{d^2y}{dx^2}$

$$\begin{cases} x = 2t - \sin 2t \\ y = 1 - \cos 2t \end{cases}$$

$$\frac{dx}{dt} = 2 - 2\cos 2t$$

$$\frac{dy}{dt} = 2\sin 2t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2\sin 2t}{2 - 2\cos 2t}$$

$$= \frac{\sin 2t}{1 - \cos 2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{(1-\cos 2t)(2\cos 2t) - (\sin 2t)(2\sin 2t)}{(1-\cos 2t)^2\,(2 - 2\cos 2t)}$$

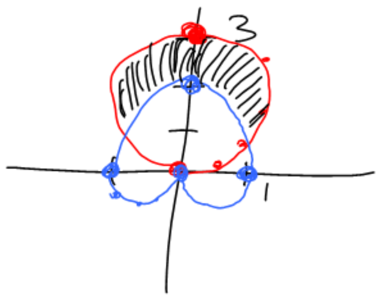
$$= \frac{2\cos 2t - 2\cos^2 2t - 2\sin^2 2t}{2(1-\cos 2t)^3}$$

$$= \frac{2\left[\cos 2t - \overset{-1}{\cancel{\cos^2 2t - \sin^2 2t}}\right]}{2\,(1-\cos 2t)^3}$$

$$= \frac{\cos 2t - 1}{(1-\cos 2t)^3}$$

$$= \frac{-1}{(1-\cos 2t)^2}$$

Ex: Find the area inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$.



Intersection

$$r = r$$

$$3\sin\theta = 1+\sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

By symmetry, A = double the area from $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2\,d\theta - 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2\,d\theta$$

$$\underbrace{}_{A_1} \qquad \underbrace{}_{A_2}$$

$$A_1 = \int_{\pi/6}^{\pi/2} 9\sin^2\theta \, d\theta$$

$$= \frac{9}{2}\int_{\pi/6}^{\pi/2} (1-\cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/2}$$

$$= \frac{9}{2}\left[\frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right]$$

$$= \frac{9}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{3\pi}{2} + \frac{9\sqrt{3}}{8}$$

$$A_2 = \int_{\pi/6}^{\pi/2} \left(1 + 2\sin\theta + \underbrace{\cancel{\sin^2\theta}}_{\frac{1}{2} - \frac{\cos 2\theta}{2}}\right) d\theta$$

$$= \left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/6}^{\pi/2}$$

$$= \left(\frac{\pi}{2} + \frac{\pi}{4}\right) - \left(\frac{\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}$$

$$= \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$$

$$A = A_1 - A_2$$

$$= \frac{3\pi}{2} + \frac{9\sqrt{3}}{8} - \left(\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right)$$

$$= \pi$$

Ex: Use 3 nonzero terms of an appropriate series to estimate

$$\int_0^{0.6} \sqrt[4]{1+x^2}\, dx$$

$$(1+x)^k \approx 1 + kx + \frac{k(k-1)}{2}x^2$$

$$(1+x)^{1/4} \approx 1 + \frac{x}{4} + \cancel{\frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)x^2} \quad \frac{-3}{32}x^2$$

$$(1+x^2)^{1/4} \approx 1 + \frac{x^2}{4} - \frac{3}{32}x^4$$

$$\int_0^{0.6} \sqrt[4]{1+x^2}\, dx \approx \int_0^{0.6} \left(1 + \frac{x^2}{4} - \frac{3}{32}x^4\right) dx$$

$$\approx \left. x + \frac{x^3}{12} - \frac{3x^5}{160} \right|_0^{0.6}$$

$$\approx 0.62$$